# ON A NONLINEAR THEORY OF ELEMENTARY PARTICLES

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**Abstract**

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# ON A NONLINEAR THEORY OF ELEMENTARY PARTICLES

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Let us consider a nonlinear generalization of the Dirac equation ([1–2]) for a quantized spinor field. To form invariants one may use our earlier method ([2]) of "contraction," which consists in making all spinor fields dimensionless in the expression for the interaction of two pairs of fermions, for example in the theory of $\beta$-decay. This method is based on the idea of a universal Fermi interaction and of describing matter by a single "world spinor." Then, restricting ourselves to the fulfillment of the conditions of the ordinary theory and not introducing derivatives into the nonlinear terms, the Lagrangian may be written in the form ([2])*

$$L = L_0 + \frac{1}{2}\sum_m \{g_m[(\bar{\psi}O_m\psi)^2 + (\psi O_m^{\mathrm{Tr}}\bar{\psi})^2] +$$

$$+ g'_m[(\bar{\psi}O_m\psi), (\psi O_m^{\mathrm{Tr}}(\bar{\psi})]_+ + g''_m[(\psi c O_m\psi), (\bar{\psi}O_m c\bar{\psi})]_+\}, \qquad (1)$$

where $O_m = 1,\ \gamma_\mu,\ \gamma_\mu\gamma_\nu,\ \gamma_5\gamma_\mu,\ \gamma_5$. If parity is not conserved, each $O_m$ may be multiplied by $(1\pm\alpha\gamma_5)$. As F. Lörst emphasized, in the unquantized theory there is a number of relations among the various terms of (1). In view of the absence of derivatives in the nonlinear terms, we consider it expedient to preserve the usual three-dimensional anticommutation relations for free fields ([3]). Taking the anticommutation relations into account makes it possible to reduce part of the nonlinear summands (1) to others and to show that some of these summands are identically equal to zero (for example, for $O_m = \gamma_\mu\gamma_\nu,\ \gamma_5\gamma_\nu$ in the third square bracket of (1)). The number of independent constants is reduced still further if one requires that the nonlinear part of the Lagrangian be invariant under Lorentz transformations that commute with the Pauli transformations ([4]), which are realized in $\beta$-decay for the neutrino:

$$\left.\begin{array}{c} \psi \to a\psi + bi\gamma_5\psi^c, \\ i\gamma_5\psi^c \to -b^*\psi + a^*i\gamma_5\psi^c \end{array}\right\} \quad \text{for } aa^* + bb^* = 1,\ \psi^c = i\gamma_5\gamma_2\psi^*. \qquad (2)$$

*The following notation is used:

$$\gamma_i = \begin{vmatrix} 0 & i\sigma_i \\ i\sigma_i & 0 \end{vmatrix} \quad (i=1,2,3); \qquad \gamma_0 = \begin{vmatrix} 0 & iE_2 \\ -iE_2 & 0 \end{vmatrix}; \qquad \gamma_5 = \begin{vmatrix} -iE_2 & 0 \\ 0 & +iE_2 \end{vmatrix};$$

$$\bar{\psi} = \psi^* \gamma_0; \qquad c = \gamma_5 \gamma_0 \gamma_2; \qquad a_\mu b_\mu = -a_i b_i + a_0 b_0;$$

a spinor standing to the left of a matrix is regarded as transposed. We emphasize here that the quadratic terms ([2]) obtained by us earlier in the third square bracket (1) could have been obtained from the theory of $\beta$-decay, in which baryon and lepton charges are not conserved. We note that Heisenberg and Pauli have recently used part of the invariants indicated in ([2]), namely the square of the pseudoscalar.

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Commutation with reflection takes place only for one of the two choices of coefficients: $\pm 1$ or $\pm i$ for the corresponding matrices that define the reflection.

These transformations are isomorphic, as is easy to see, to the spinor representation of the rotation group. To select invariants, introduce the 8-component spinor

$$\Psi = \begin{pmatrix} \psi \\ i\gamma_5 \psi^c \end{pmatrix},$$

whose transformations under Lorentz transformations are specified by the matrices $\gamma_\mu \times E_2$, and under the transformations (2) by the matrices $E_4 \times i\rho_i$, where $\rho_i$ are the Pauli matrices. Supplementing (2) by the reflection of the four coordinates ($\psi \to \gamma_5 \psi$), we extend (2) to a unitary group.[4] Introduce seven anticommuting matrices $\Gamma_r$ $(r = 1, 2, \dots, 7)$:

$$\Gamma_{\mu+1} = \gamma_\mu \times E_2 \quad (\mu = 0,1,2,3); \qquad \Gamma_{i+4} = \hat{\tau}_i = i\gamma_5 \times \rho_i \quad (i = 1,2,3). \quad (3)$$

For comparison of (2) with isotopic rotations, we rewrite the spinor $\Psi$ in a form in which, according to Feynman [5], each particle is specified by a two-component semispinor

$$\Psi \equiv \frac{1}{2} \begin{pmatrix} (1+i\gamma_5)\psi + (1-i\gamma_5)\psi^c \\ (1+i\gamma_5)\psi^c - (1-i\gamma_5)\psi \end{pmatrix} = \begin{pmatrix} \psi_p \\ \psi^c_{\bar{n}} \\ \psi_n \\ -\psi^c_{\bar{p}} \end{pmatrix}. \quad (4)$$

The matrices $\Gamma_{i+4} = \hat{\tau}_i = i\gamma_5 \times \rho_i$ in this case carry out transformations* in isotopic space, where the isotopic two-component spinor consists of proton—antineutron (respectively antiproton—neutron). The fastest way of constructing combinations of $\Psi$ invariant with respect to Lorentz transformations and (2) is to consider rotations in a 6-dimensional space** with basis matrices $\Gamma_{\mu+1} = \gamma_\mu \times E_2$ ($\mu = 0, 1, 2, 3$) and $\Gamma_{i+4} = \hat{\tau}_i = i\gamma_5 \times \rho_i$ ($i = 1, 2$), where the product of the indicated six matrices is equal to $\Gamma_7 = \hat{\tau}_3 = i\gamma_5 \times \rho_3$. First of all, note that the spinor $\Psi$ is self-conjugate (real). Indeed, forming the product of all the real matrices $C = \gamma_2 \times i\rho_2$, it is easy to verify that

$$\Psi^c = +C\Psi^* = \Psi. \tag{5}$$

Thus, invariants of the type $\Psi^+\mathcal{E}\Omega_m\Psi$, where $\mathcal{E}$ is the product of Hermitian matrices $\mathcal{E} = i\gamma_0 \times \rho_3$ from among the six basis matrices and $\Omega_m = 1, \Gamma_r, \Gamma_r\Gamma_p$, etc., $r, p = 1, \dots, 6$, reduce*** to invariants of the type $\Psi C\mathcal{E}\Omega_m\Psi$. Further, owing to the anticommutation of the variation $\delta\Psi_\alpha$ with $\Psi_\beta$, in the field equations only those invariants will give a nonvanishing contribution which satisfy the condition****

$$(C\mathcal{E}\Omega_m)^{\mathrm{Tr}} = -C\mathcal{E}\Omega_m, \tag{6}$$

since

$$(C\mathcal{E}\Omega_m)_{\pi\sigma}(\delta\Psi_\pi\Psi_\sigma + \Psi_\pi\delta\Psi_\sigma) = [C\mathcal{E}\Omega_m - (C\mathcal{E}\Omega_m)^{\mathrm{Tr}}]_{\pi\sigma}\delta\Psi_\pi\Psi_\sigma.$$

The condition (6) is satisfied by $\Omega_m = 1, \Gamma_r, \Gamma_r\Gamma_p$. We restrict ourselves to consideration of the following nonlinear part of the Lagrangian ($g_v$ has the dimension of the Fermi constant):

$$L_n = g_v\left[(\Psi^+\mathcal{E}\Gamma_r\Psi)^2 + (\Psi^+\mathcal{E}\Gamma_r\Gamma_7\Psi)^2\right], \tag{7}$$

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* Since the simultaneous transformation $\psi_p\psi_{\bar{N}}^c$ and $\psi_{\bar{p}}^c\psi_N$ is performed, then, owing to the imaginary character of $\hat{\tau}_i$, we have $\hat{\tau}_i\hat{\tau}_j = i\gamma_5 \times e_{ijk}\hat{\tau}_k$, and $\hat{\tau}_i$ cannot simply be identified with the isotopic matrices.

** Invariance with respect to mixed rotations of the form $\gamma_\mu \times \hat{\tau}_i = i\gamma_5\gamma_\mu \times \rho_i$ ($i = 1, 2$) is equivalent to invariance with respect to the product of reflection in the usual sense and charge conjugation (strong inversion).

*** In the general case, in the indicated combined space one may also introduce non-self-conjugate spinors.

**** Always $(C\mathcal{E}\Omega_m)^{\mathrm{Tr}} = \pm C\mathcal{E}\Omega_m$.

which turns into zero* for $\Psi$ satisfying the condition (neutrino):

$$\Psi^* = \pm\Gamma_7\Psi. \tag{8}$$

The Lagrangian (7) can be rewritten in a form in which the vanishing for particles of type (8) is evident:

$$L_n = \frac{1}{2}g_v\,[\Psi^+\gamma_0\rho_3\Gamma_r(1+\Gamma_7)\Psi,\ \Psi^+\gamma_0\rho_3\Gamma_r(1-\Gamma_7)\Psi]_+, \tag{9}$$

Taking into account that $(1\pm i\gamma_5)\gamma_0(1\pm i\gamma_5)=0$, $(1\mp\hat{\tau}_3)(1\pm\hat{\tau}_3)=0$, instead of (9) one may write

$$\begin{aligned} L_n = &\frac{1}{2}g_v\,[\Psi^+\gamma_0\rho_3\gamma_\mu(1+\hat{\tau}_3)\Psi,\ \Psi^+\gamma_0\rho_3\gamma_\mu(1-\hat{\tau}_3)\Psi]_+ \\ &+\frac{1}{4}g_v\,[\Psi^+\gamma_0\rho_3(1-\hat{\tau}_3)\hat{\tau}_i(1+\hat{\tau}_3)\Psi,\ \Psi^+\gamma_0\rho_3(1+\hat{\tau}_3)\hat{\tau}_i(1-\hat{\tau}_3)\Psi]_+, \end{aligned} \tag{9a}$$

$$\mu = 0,\dots,3, \qquad i=1,2,3.$$

Varying (9) with respect to $(1\mp\hat{\tau}_3)\gamma_0\rho_3\Psi^* = (1\mp\hat{\tau}_3)\gamma_0\rho_3\gamma_2\tau_2\Psi$, we obtain the field equations:

$$\begin{aligned} &D_0(1\mp\hat{\tau}_3)\Psi(x) + \frac{1}{2}g_v[\gamma_\mu(1\mp\hat{\tau}_3)\Psi(x),\ \Psi^+(x)\gamma_0\rho_3\gamma_\mu(1\mp\hat{\tau}_3)\Psi(x)]_+ \\ &\quad+\frac{1}{2}g_v[\hat{\tau}_i(1\mp\hat{\tau}_3)\Psi(x),\ \Psi^+(x)\gamma_0\rho_3(1+\hat{\tau}_3)\hat{\tau}_i(1\mp\hat{\tau}_3)\Psi(x)]_+ = 0, \end{aligned} \tag{10}$$

where $D_0$ is the usual Dirac operator.

Let us now pose the problem of constructing, for the nonlinear Lagrangian chosen, equations for the causal propagation functions (Green' s functions). It is convenient here to define the latter quantity** in the following way, not including the matrices $\delta = \gamma_0\times\rho_3$:

$$G^{\pm}_{\hat{\alpha}\hat{\beta}}(x,x') = i\langle T((1\pm\hat{\tau}_3(\Psi_{\hat{\alpha}}(x),\Psi^*_{\hat{\beta}}(x'))))\rangle_0. \tag{11}$$

Here $\hat{\alpha}=\alpha\times\alpha'$, $\hat{\beta}=\beta\times\beta'$; $\beta=1,2,3,4$; $\alpha',\beta'=1,2$. (In those cases when $\alpha'=\beta'$, we shall write simply $\alpha,\beta$ instead of $\hat{\alpha},\hat{\beta}$.) To solve the problem posed of constructing equations, we introduce in addition into the Lagrangian terms with fictitious sources $J^{\pm}(x)$, commuting with $\Psi$, in the following way:

$$\Delta L=\frac{1}{2}\sum_{(\pm)}\left[J_\mu^{\pm}(x)\Psi^+(x)\gamma_0\rho_3\gamma_\mu(1\pm\hat\tau_3)\Psi(x)+J_i^{\pm}(x)\Psi^+\gamma_0\rho_3\hat\tau_i(1\pm\hat\tau_3)\Psi\right], \tag{12}$$

$$\mu=0,\dots,3,\qquad i=1,2,3.$$

Multiplying (10) by $\Psi^*_\beta(x')$ on the right and on the left, and then proceeding in the usual manner ([7]), from (10) we obtain, for $J=0$, for $G^+$

$$-i\gamma_0\gamma_\mu\frac{\partial G^+(x,x')}{\partial x^\mu}+g_v\gamma_0\gamma_\mu\frac{\delta G^+(x,x')}{\delta J_\mu^-(x)}+ig_v\gamma_0\gamma_\mu G^+(x,x')\ \mathrm{tr}\,\gamma_\mu\gamma_0\rho_3G^-(x,x)$$
$$+g_v\gamma_0\gamma_5\rho^i\frac{\delta G^+(x,x')}{\delta J_i^-(x)}+ig_v\gamma_0\gamma_5\rho_iG^+(x,x')\ \mathrm{tr}\,\rho_i\gamma_5\gamma_0\rho_3G^-(x,x)=\delta(x-x'). \tag{13}$$

Here we consider the mass in the operator $D_0$, following Heisenberg, to be equal to zero ([1]), and the operation $T(A(x),B(x'))$ for $x_0=x_0'$ is understood as the half-sum pre-

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* The combination (7) is the only one satisfying, besides the indicated conditions, the requirement of invariance with respect to Salam–Touchek transformations ([6]), which, in the notation adopted, are written in the form $\Psi\to e^{i\alpha\Gamma_7}\Psi$.

** Obviously, $(G^+\gamma_0)_{\alpha\beta}$ coincides with the standard definition of the proton causal Green' s function. The cases $\alpha'=\beta'=1$ and $\alpha'=\beta'=2$ lead to equivalent expressions (in the absence of isotopic transitions) owing to the charge invariance of the vacuum. Let us also note that, in order to preserve invariance of the Lagrangian, $J_\mu^{\pm}$ must change sign under charge conjugation.

cases as $x_0-x_0'\to\pm0$. The matrices $\gamma_0\gamma_\mu$ entering into (13), unlike $\gamma_0\gamma_5$, contain no transitions from $\psi_p$ to $\psi_n$. An analogous equation is obtained for $G^-(x,x')$. Owing to charge symmetry, in (13) one may restrict oneself to the case $\alpha'=1$. Let us introduce the mass operator $M$, taking into account the translational invariance of the Green functions:

$$\int M^+(x,x'')\gamma_0G^+(x'',x')\,dx''=\frac{ig_v}{2\pi^2}\left[\left(\mathrm{tr}\int\gamma_0\rho_3\gamma_\mu G^-(k)\,dk\right)\gamma_0\gamma_\mu+\right.$$
$$\left.+\left(\mathrm{tr}\int\gamma_0\rho_3\gamma_5\rho_iG^-(k)\,dk\right)\gamma_0\gamma_5\right]G^+(x,x')+$$
$$+g_v\gamma_0\gamma_\mu\frac{\delta G^+(x,x')}{\delta J_\mu^-(x)}+g_v\gamma_0\gamma_5\rho_i\frac{\delta G^+(x,x')}{\delta J_i^-(x)}. \tag{14}$$

With its aid, (13) is rewritten in the form

$$-i\gamma_0\gamma_\mu \frac{\partial G^+(x,x')}{\partial x^\mu} + \int M^+(x,x'')\gamma_0 G^+(x'',x')\,dx'' = \delta(x-x'). \tag{15}$$

Thus, the equation for the causal Green function in the nonlinear case, in the absence of external fields, turns out to be equivalent to the equation for the linear case in the presence of interaction with a bosonic vacuum of the "electromagnetic" and "mesonic" type, with a field mass and a "bare" mass; moreover, the latter is specified by the coefficient before $G^+$ in the first term on the right-hand side of (14) and is determined, ultimately, by the values of the Green functions on the light cone. The indicated bosonic fields may be connected with the sources $J^\pm$ by the usual relations and introduced into the Lagrangian. We note that, while preserving invariance only with respect to rotations in ordinary and isotopic space separately, one may regard the coefficients $g_v$ in the "electromagnetic" and "mesonic" terms as different and set, for example, $J^- \equiv 0$.

The result obtained leads to the conclusion, proved by us earlier (?) in the first approximation of perturbation theory, that the addition of nonlinear terms in the case of a free particle reduces to an effective change (or introduction) of the mass. Continuing the indicated reasoning, one may obtain that, in the case of a two-particle Green function, the equations of the nonlinear theory will be externally indistinguishable from the equations of the linear theory in the presence of a bosonic vacuum*. In this case the square of the effective coupling constant and the mass will be proportional to a coefficient containing $g_v$. It is also essential that the bosonic fields can be interpreted, according to the preceding, in the nonlinear theory as the result of unification** of "primary" spinors in the spirit of "fusion."

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* The interaction of the Fermi type begins to manifest itself directly only in the four-particle Green function (cf. (?)).

** From the indicated peculiar "equivalence theorem" of nonlinearity and the bosonic vacuum there also follows the possibility of transferring into the nonlinear theory a number of results of the quantum theory of interacting fields and, in particular, results of the theory of renormalizations.

*Note: Figure translations are in progress. See original paper for figures.*