MATHEMATICS

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1958

SovietRxiv

Abstract

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THE UNSOLVABILITY OF THE HOMEOMORPHY PROBLEM

1. We shall call the **homeomorphy problem** the problem of finding an algorithm that recognizes, for any two given polyhedra, whether they are homeomorphic. Here the polyhedra are specified combinatorially by their triangulations, which makes it possible to understand the term “algorithm” here in its exact sense, i.e., for example, as a “normal algorithm” ([3]).

Alongside the general homeomorphy problem, various particular homeomorphy problems naturally arise, relating to polyhedra of one class or another. For example, fixing a natural number n, one may pose the homeomorphy problem for polyhedra of dimension not exceeding n. One may also pose the homeomorphy problem for n-dimensional manifolds, if one agrees on a definite understanding of the term “manifold.”

Another natural restriction imposed on the polyhedra being compared is the fixing of one of them. This gives rise to the problem of homeomorphy to a given polyhedron A, consisting in finding an algorithm that recognizes, for any polyhedron, whether it is homeomorphic to the polyhedron A.

Some of these problems have long been solved, for example, the homeomorphy problem for 2-dimensional manifolds or the problem of homeomorphy to a given 2-dimensional manifold. However, the following theorem holds.

Theorem 1. *For every natural number n greater than three, one can specify an n-dimensional manifold M^n such that the problem of homeomorphy of manifolds to the manifold M^n is unsolvable.*

We understand the term “manifold” here in the sense of Poincaré ([4]) and Veblen ([5]).

Corollary 1. *The homeomorphy problem for n-dimensional manifolds is unsolvable for $n > 3$.*

Corollary 2. *The homeomorphy problem for polyhedra of dimension not exceeding n is unsolvable for $n > 3$.*

Corollary 3. *The general homeomorphy problem is unsolvable.*

2. The proof of Theorem 1 is based on the following construction. For every natural number r we consider the $2r$-letter alphabet

$$\Gamma_r = \{\alpha_1^1, \dots, \alpha_r^1, \alpha_1^{-1}, \dots, \alpha_r^{-1}\}.$$

For every r and every system of words P_i $(i = 1, \dots, m)$ in the alphabet Γ, a 4-dimensional manifold $\mathfrak{M}(P_1 * \dots * P_m * r)$ is constructed in such a way that the following conditions are satisfied.

1. The fundamental group of the manifold $\mathfrak{M}(P_1 * \dots * P_m * r)$ is isomorphic to the group defined by the system of relations

$$P_i \leftrightarrow \Lambda \qquad (i = 1, \dots, m) \tag{1}$$

between the generating elements $\alpha_1^1, \dots, \alpha_r^1$. The letters $\alpha_1^{-1}, \dots, \alpha_r^{-1}$ are regarded here as elements inverse to the elements $\alpha_1^1, \dots, \alpha_r^1$.*

* The group calculus corresponding to this group (see ([3]), p. 341) in the alphabet Γ_r can be defined by the system of relations obtained from system (1) after adjoining the relations $\alpha_i^\varepsilon \alpha_i^{-\varepsilon} \leftrightarrow \Lambda$ $(i = 1, \dots, r;\ \varepsilon = \pm 1)$.

U2. The manifolds $\mathfrak{M}(P_1 * \dots * P_m * r)$ and $\mathfrak{M}(Q_1 * \dots * P_m * r)$ are homeomorphic if the system of words $Q_1 * \dots * Q_m$ is obtained from the system of words $P_1 * \dots * P_m$ as the result of substituting the empty word in place of an occurrence of the word $\alpha_i^\varepsilon \alpha_i^{-\varepsilon}$ $(i = 1, \dots, r;\ \varepsilon = \pm 1)$.

U3. The manifolds $\mathfrak{M}(P_1 * \dots * P_m * r)$ and $\mathfrak{M}(Q_1 * \dots * Q_m * r)$ are homeomorphic if, among the numbers $1, \dots, m$, there is a number i such that Q_i is the result of a cyclic permutation of the letters in the word P_i and such that

$$Q_j = P_j \tag{2}$$

for $1 \leqslant j \leqslant m$ and $j \neq i$.

U4. The manifolds $\mathfrak{M}(P_1 * \dots * P_m * r)$ and $\mathfrak{M}(Q_1 * \dots * Q_m * r)$ are homeomorphic if, among the numbers $1, \dots, m$, there is a number i such that Q_i is the group inverse of the word P_i and such that, for $1 \leqslant j \leqslant m$ and $j \neq i$, equality (2) holds.

The **group inverse** of a word P in the alphabet Γ_r is defined here as the word obtained from P as a result of reversing the order of the letters, followed by replacing each letter α_i^ε by the letter $\alpha_i^{-\varepsilon}$.

U5. The manifolds $\mathfrak{M}(P_1 * \dots * P_m * r)$ and $\mathfrak{M}(Q_1 * \dots * Q_m * r)$ are homeomorphic if, among the numbers $1, \dots, m$, there are numbers i and h such that $i \neq h$,

$$Q_i = P_{iP} h$$

and such that, for $1 \leqslant j \leqslant m$ and $j \neq i$, equality (2) holds.

U6. The manifolds $\mathfrak{M}(*\alpha_1^k * \ldots * \alpha_r^k * r)$ and $\mathfrak{M}(*^k 0)$ are homeomorphic, whatever the natural number k.

The construction of the manifold $\mathfrak{M}(P_1 * \ldots * P_m * r)$ can be carried out as a certain refinement of the construction, indicated by Seifert and Threlfall, of a 4-dimensional manifold with a prescribed fundamental group (see (2), p. 208).

Lemma 1. If the group with generators $\alpha_1^1, \ldots, \alpha_r^1$, defined by the system of relations

$$R_i \leftrightarrow \Lambda \quad (i = 1, \ldots, k) \tag{3}$$

in the alphabet Γ_r, is trivial, then the manifold $\mathfrak{M}(R_1 * \ldots * R_k * r + 1|r)$ is homeomorphic to the manifold $\mathfrak{M}(*^k 0)$.

Lemma 2. If the group referred to in Lemma 1 is not trivial, then the manifold $\mathfrak{M}(R_1 * \ldots * R_k * r + 1|r)$ is not homeomorphic to the manifold $\mathfrak{M}(*^k 0)$.

Now fix natural numbers r and k. We shall consider groups with generators $\alpha_1^1, \ldots, \alpha_r^1$, defined by all possible systems of k relations (3) in the alphabet Γ_r. We shall call such groups (r, k)-groups.

It follows from Lemmas 1 and 2 that, with the aid of any algorithm recognizing, for any manifold, whether it is homeomorphic to the manifold $\mathfrak{M}(*^k 0)$, one can construct an algorithm recognizing, for any (r, k)-group, whether it is trivial. Meanwhile, from the construction carried out by S. I. Adian in (1), it follows that the numbers r and k can be chosen so that no algorithm recognizing the triviality of an (r, k)-group will be possible. Take such a pair of numbers (r, k) and put

$$M^4 = \mathfrak{M}(*^k 0).$$

Then the problem of homeomorphism of manifolds to the 4-dimensional manifold M^4 will be undecidable.

Finally, it is not difficult to see that, for every natural number n greater than four, the problem of homeomorphism to the n-dimensional manifold

$$M^n = M^4 \times S^{n-4},$$

where S^h denotes the h-dimensional sphere, is undecidable.

This completes the proof of Theorem 1.

3. Analogously to the problems of homeomorphy, problems of homotopy equivalence may be posed. Their formulations are obtained from the formulations of the problems of homeomorphy by replacing the words

"homeomorphic," "homeomorphism," "homeomorphically" by the words "homotopy equivalent," "homotopy equivalence," "up to homotopy equivalence." But such a replacement is, obviously, permissible in Lemmas 1 and 2. This gives the following results.

Theorem 2. *For every natural number n greater than three, the problem of homotopy equivalence of manifolds to the manifold M^n is unsolvable.*

Corollary 1. *The problem of homotopy equivalence of n-dimensional manifolds is unsolvable for $n > 3$.*

Corollary 2. *The problem of homotopy equivalence of polyhedra of dimension not exceeding n is unsolvable for $n > 3$.*

Corollary 3. *The general problem of homotopy equivalence is unsolvable.*

Received
21 III 1958

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Note: Figure translations are in progress. See original paper for figures.