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THE ERROR OF DIRECT METHODS IN THE NONLINEAR THEORY OF SHELLS

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Abstract

Full Text

THEORY OF ELASTICITY

I. I. Vorovich

**THE ERROR OF DIRECT METHODS IN THE
NONLINEAR THEORY OF SHELLS**

(Presented by Academician V. I. Smirnov, 12 V 1958)

In ⁽¹⁾ an existence theorem is given for solutions of the system

$$B \left\{ \frac{1}{AB} [(Bu)_\alpha + (Av)_\beta] \right\}_\alpha + \frac{1-\nu}{2} A \left\{ \frac{1}{AB} [(Au)_\beta - (Bv)_\alpha] \right\}_\beta + \frac{1-\nu}{R_1 R_2} ABu = f_1\{w\}; \quad (1)$$

$$A \left\{ \frac{1}{AB} [(Av)_\beta + (Bu)_\alpha] \right\}_\beta + \frac{1-\nu}{2} B \left\{ \frac{1}{AB} [(Bv)_\alpha - (Au)_\beta] \right\}_\alpha + \frac{1-\nu}{R_1 R_2} ABv = f_2\{w\}; \quad (2)$$

$$\nabla^4 w = f_3\{u, v, w\}; \quad (3)$$

$$u|_{\Gamma_\Omega} = v|_{\Gamma_\Omega} = 0; \quad (4)$$

$$w|_{\Gamma_\Omega} = \frac{dw}{dn} \Big|_{\Gamma_\Omega} = 0, \quad (5)$$

which, under certain conditions, describes large deformations of a shell. In (1)–(5), u, v, w are the displacements of points of the middle surface of the shell; A, B are the coefficients of the first fundamental form of the middle surface.

For the approximate solution of the boundary-value problem (1)–(5), direct methods are used either in the form of P. F. Papkovich, or in the form of Kh. M. Mushtari. Let us first consider the method of P. F. Papkovich. Suppose that u, v are expressed from (1)–(2) in terms of w , as a result of which (3) becomes an equation with respect to w . Equation (3) can be replaced by the equation $w - Gw = 0$, where G is a certain completely continuous operator acting in the space $H_{1\Omega}$ of functions w satisfying conditions (5), and in which the norm is defined by the bending energy of the shells. We shall seek an approximate solution of the problem w_n in the form

$$w_n = \sum_{k=1}^n a_{nk} \chi_k(P); \quad a_{nk} = \int_{\Omega} f_3\{w_n\} \chi_k(P) AB d\alpha d\beta, \quad (6)$$

where $\chi_k(P)$ is an orthonormal basis in $H_{1\Omega}$.

The following facts hold, justifying the application of the method of P. F. Papkovich in the present problem:

Theorem 1. *Let:*

- 1) *The middle surface of the shell Σ be given by the equation $\mathbf{r} = \mathbf{r}(\alpha, \beta)$, where α, β vary in a bounded plane domain Ω .*
- 2) *\mathbf{r} have continuous second derivatives in $\bar{\Omega}$.*
- 3) *The boundary Ω, Γ_{Ω} , can be divided into a finite number of arcs on each of which the tangent varies continuously.*
- 4) *The external forces $X, Y, Z \in L_p, p > 1$.*

In this case:

- 1) *The system of equations (6) coincides with the system*

$$\frac{\partial J(w_n)}{\partial a_{nk}} = 0,$$

where J is the functional of the total energy of the system shell–external forces.

- 2) *The system (6) has real solutions for every $n \geq 1$.*
- 3) *The entire set of approximations $\{w_n\}, n = 1, 2, \dots$, is strongly compact in $H_{1\Omega}$.*
- 4) *Every weakly convergent sequence in $H_{1\Omega}$ from $\{w_n\}$ converges strongly, and every strong limit of sequences from $\{w_n\}$ is a solution of the equation $w - Gw = 0$.*
- 5) *Let w^0 be a strict relative minimum (maximum)* of the functional J or a solution of the equation $w - Jw = 0$ having nonzero index ⁽³⁾. In this case w^0 is a limit point in $H_{1\Omega}$ of the set of approximations $\{w_n\}$.*

In some cases it proves possible to estimate the rate of convergence of sequences from $\{w_n\}$. These estimates are based on certain data on the degree of smoothness of the solutions of our problem, which are given by the following theorem.

Theorem 2.** Suppose conditions 1), 4) of Theorem 1 are satisfied. In this case every solution of the equation $w - Gw = 0$ has the following properties:

- 1) *If $\Sigma \in H(4, m, 0); \Gamma_{\Omega} \in \mathcal{L}_4(B_1, 0)$, then*

$$u, v \in \begin{cases} H\left(1, B_2, \frac{p-2}{p}\right), & p > 2, \\ W_{\frac{2p}{2-p}}^{(1)}, & p \leq 2, \end{cases} \quad w \in \begin{cases} H'\left(3, B_3, \frac{p-2}{p}\right), & p > 2, \\ W_{\frac{2p}{2-p}}^{(3)}, & p \leq 2. \end{cases} \quad (7)$$

2) If $\Sigma \in H(k+4, m, 0)$; $\Gamma_\Omega \in \mathcal{L}_{4+k}(B_4, \lambda)$; $X, Y, Z \in H(k, B_5, \lambda)$, then

$$u, v \in \begin{cases} H(2+k, B_6, \lambda), & \lambda < 1, \\ H(2+k, B_6, 1-\varepsilon), & \lambda = 1, \end{cases} \quad w \in \begin{cases} H(4+k, B_7, \lambda), & \lambda < 1, \\ H(4+k, B_7, 1-\varepsilon), & \lambda = 1. \end{cases} \quad (8)$$

In (7), (8), $\varepsilon > 0$ is an arbitrarily small number; B_i , $i = 1, \dots, 7$, are certain constants.

Suppose now that χ_k are defined by the relation

$$\chi_k(\alpha, \beta) = \varphi^2(\alpha, \beta)\Pi_k(\alpha, \beta), \quad (9)$$

where $\varphi(\alpha, \beta) = 0$ is the equation of Γ_Ω , and $\Pi_k(\alpha, \beta)$ is a polynomial in α, β . In this case, on the basis of Theorem 2 and certain results from ^(3, 5), the following may be formulated:

Theorem 3. Suppose conditions 1), 4) of Theorem 1 are satisfied. Suppose, further, that $\Sigma \in H(4+k, m, 0)$; $\Gamma_\Omega \in \mathcal{L}_{4+k}(B_1, \lambda)$; $X, Y, Z \in H(k, B_2, \lambda)$; $\varphi \in H(k+4, B_3, 1)$; $\text{grad } \varphi|_{\Gamma_\Omega} \neq 0$; $k \geq 0$.

* We shall call a critical point w^0 of the functional J a **strict relative minimum (maximum)** if one can specify in $H_{1\Omega}$ such a closed ball with center at w^0 in which the strict inequality $J(w) > J(w^0)$ ($J(w) < J(w^0)$) holds.

** The classes of functions $H(k, m, \lambda)$ and the classes of surfaces $\mathcal{L}_k(m, \lambda)$ are introduced in ⁽⁴⁾.

If w_n are determined by means of (6), χ_k by means of (9), and w^0 is a nonexceptional solution of (6), then from $\{w_n\}$ one can choose a sequence of approximations such that

$$\|w^0 - w_n\|_{H_{1\Omega}} \leq \begin{cases} \frac{m}{n^{\frac{2+k+\lambda}{2}}}, & \lambda < 1; \\ \frac{m}{n^{\frac{3+k-\varepsilon}{2}}}, & \lambda = 1 \text{ and } \varepsilon > 0 \text{ arbitrarily small.} \end{cases} \quad (10)$$

Let us now consider the method of Kh. M. Mushtari. For this purpose we introduce the space $H_{2\Omega}$ of vector-functions $\vec{\omega}^*(u, v)$ satisfying the boundary

conditions (4), the norm in which is determined by the potential energy of stretching of the shell ^(1,2). In addition, we introduce the space $H_{3\Omega}$ of vector-functions $\vec{\omega}(u, v, w)$ such that $\vec{\omega}^*(u, v) \in H_{2\Omega}$, $w \in H_{1\Omega}$, putting $\|\vec{\omega}\|_{H_{3\Omega}}^2 = \|\vec{\omega}^*\|_{H_{2\Omega}}^2 + \|w\|_{H_{1\Omega}}^2$. It can be shown that the system (1)–(5) reduces to the operator equation $\vec{\omega} = C\vec{\omega}$, where C is a completely continuous operator in $H_{3\Omega}$.

We shall seek an approximate solution of our problem in the form

$$w_n = \sum_{k=1}^n a_{nk} \chi_k(P); \quad \vec{\omega}_n^*(u_n, v_n) = \sum_{k=1}^n c_{nk} \vec{b}_k(\varphi_k, \psi_k), \quad (11)$$

where $\chi_k, \varphi_k, \psi_k$ form an orthonormal basis in $H_{3\Omega}$. The constants a_{nk}, c_{nk} are found from the system

$$\frac{\partial J}{\partial a_{nk}} = 0; \quad \frac{\partial J}{\partial c_{nk}} = 0; \quad k = 1, \dots, n. \quad (12)$$

The following facts hold, justifying the application of the method of Kh. M. Mushtari.

Theorem 4. Suppose that conditions 1)–4) of Theorem 1 are satisfied. In this case:

- 1) System (12) has real solutions for every $n \geq 1$.
- 2) The whole set $\{\vec{\omega}_n(u_n, v_n, w_n)\}$ of approximations is strongly compact in $H_{3\Omega}$.
- 3) Every weakly convergent sequence in $H_{3\Omega}$ from $\{\vec{\omega}_n(u_n, v_n, w_n)\}$ converges strongly, and every limit of elements from $\{\vec{\omega}_n(u_n, v_n, w_n)\}$ gives a solution of the equation $\vec{\omega} = C\vec{\omega}$.
- 4) Let $\omega^0(u^0, v^0, w^0)$ be a strict relative minimum (maximum) of the functional J in $H_{3\Omega}$, or a solution of the equation $\vec{\omega} - C\vec{\omega} = 0$ with nonzero index ⁽³⁾. In this case $\vec{\omega}^0$ is a limit point in $H_{3\Omega}$ of the set $\{\vec{\omega}_n(u_n, v_n, w_n)\}$.

Suppose, further, that $\varphi_k, \psi_k, \chi_k$ are taken in the form $\varphi_k = \varphi \Pi_{1k}$, $\psi_k = \varphi \Pi_{2k}$; $\chi_k = \varphi^2 \Pi_{3k}$, where $\Pi_i, i = 1, 2, 3$, are polynomials. In this case the following proves valid:

Theorem 5. Suppose that the conditions of Theorem 3 hold, and that $\omega^0(u^0, v^0, w^0)$ is a nonexceptional solution ⁽⁶⁾. In this case, from $\{\vec{\omega}_n(u_n, v_n, w_n)\}$ one can choose a sequence such that

$$\|\vec{\omega}^0 - \vec{\omega}_n\|_{H_{3\Omega}} \leq \begin{cases} \frac{m}{n^{\frac{k+2+\lambda}{6}}}, & \lambda < 1, \\ \frac{m}{n^{\frac{k+3-\varepsilon}{6}}}, & \lambda = 1 \text{ and } \varepsilon > 0 \text{ arbitrarily small.} \end{cases} \quad (13)$$

Analogous results can also be formulated in the case when direct methods are applied to the system of equations with a stress function. In this case one obtains a justification of the modification of the method proposed by P. F. Papkovich and V. Z. Vlasov.

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References Cited

1. I. I. Vorovich, DAN, 117, No. 2 (1957).
2. S. G. Mikhlin, *Variational Methods in Mathematical Physics*, 1957.
3. M. A. Krasnosel' skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, 1956.
4. N. M. Günter, *Potential Theory and Its Application to the Basic Problems of Mathematical Physics*, 1953.
5. I. Yu. Kharrik, DAN, 80, No. 1 (1951).
6. I. I. Vorovich, DAN, 105, No. 1 (1955).

Note: Figure translations are in progress. See original paper for figures.

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