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Abstract

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ASYMPTOTICS OF THE SOLUTION OF A DIFFERENTIAL EQUATION OF PARABOLIC TYPE WITH A SMALL PARAMETER

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1. Let us consider the behavior, as $\varepsilon \to 0$ $(\varepsilon > 0)$, of the solution $u^\varepsilon(x,t)$ of the following Cauchy problem:

$$L_\varepsilon u^\varepsilon(x,t) = \varepsilon \sum A_{ij}(x,t)\frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} + \sum B_j(x,t)\frac{\partial u^\varepsilon}{\partial x_i} + C(x,t)u^\varepsilon - \frac{\partial u^\varepsilon}{\partial t} = 0, \quad (1)$$

$$u^\varepsilon(x,t)\big|_{t=0} = \Psi(x), \qquad (2)$$

where $x = (x_1, \dots, x_n)$ is a point of n-dimensional space E^n; $t \in [0,T]$; $A_{ij}(x,t)$, $B_i(x,t)$, $C(x,t)$, $i,j = 1,2,\dots,n$, are assumed to be bounded together with their derivatives up to order $2n$ with respect to all x_i, $i = 1,2,\dots,n$; moreover, $\partial A_{ij}/\partial t$, $\partial B_j/\partial t$, $i,j = 1,2,\dots,n$, exist and are bounded; at every point $(x,t) \in E^n \times [0,T]$,

$$\sum A_{ij}(x,t)\xi_i\xi_j \geq \alpha \sum \xi_i^2, \quad \alpha > 0.$$

The function $\Psi(x)$, everywhere in E^n, except for the points of a certain surface $F(x) = 0$, has bounded derivatives up to order $2n$. On the surface $F(x) = 0$ the function $\Psi(x)$ and all its derivatives up to order n have a discontinuity of the first kind (the behavior of derivatives of higher order on $F(x) = 0$ is of no interest to us). The surface $F(x) = 0$ has the same smoothness as the coefficients of equation (1); moreover, it is intersected by every straight line parallel to the x_1-axis, and in exactly one point.

Denote by $v(x,t)$ the solution of problem (1)–(2) for $\varepsilon = 0$ (we shall call it problem (1^0)–(2^0)), and by $l(x,t)$ the characteristic of equation (1^0) passing through the point (x,t). Then along each characteristic of equation (1^0) passing through the surface $F(x) = 0$ (we shall call it a characteristic of discontinuity), the solution $v(x,t)$ will be discontinuous. (We do not consider the convergence of the solution $u^\varepsilon(x,t)$, as $\varepsilon \to 0$, of problem (1)–(2), and also of other problems to be considered below, to the solution $v(x,t)$ of problem (1^0)–(2^0) at all points of continuity of the function $v(x,t)$, since it was proved in $(^1)$ even for solutions of nonlinear parabolic equations.)

Definition. The principal, with respect to ε as $\varepsilon \to 0$, part of the difference $u^\varepsilon(x,t) - v(x,t)$ will be called the "inner parabolic boundary layer."

Let us find the inner parabolic boundary layer and the asymptotics with respect to ε for $u^\varepsilon(x,t)$ in a neighborhood of any characteristic of discontinuity.

Replacing x and t, respectively, by y and τ, where τ is the length of the arc of the characteristic $l(x,t)$ between the points $(y,0)$ and (x,t), and $y = (y_1,\dots,y_n)$ are the coordinates of the point of intersection of $l(x,t)$ with the plane $t = 0$, we obtain

$$L_\varepsilon u^\varepsilon(y,\tau) \equiv \varepsilon \sum a_{ij}(y,\tau)\frac{\partial^2 u^\varepsilon}{\partial y_i \partial y_j} + c(y,\tau)u^\varepsilon - \frac{\partial u^\varepsilon}{\partial \tau} = 0, \qquad (1')$$

$$u^\varepsilon(y,\tau)\big|_{\tau=0} = \psi(y); \qquad y = (y_1,\dots,y_n) \in E^n, \quad \tau \in [0,T_1]. \qquad (2')$$

In this case the surface $F(x) = 0$ passes into the surface $f(y) = 0$, on which $\psi(y)$ is discontinuous together with all its derivatives up to order n. In a neighborhood of $f(y) = 0$ introduce local coordinates φ and ρ: $\varphi = (\varphi_1,\dots,\varphi_{n-1})$ are coordinates on the surface $f(y) = 0$, and ρ is the length of the normal at the corresponding point $(\varphi_1,\dots,\varphi_{n-1})$. Then $(1')$–$(2')$ passes into

$$L_\varepsilon u^\varepsilon(\rho,\varphi,\tau) \equiv \overline{L}_\varepsilon u^\varepsilon + \sigma(\sqrt{\varepsilon},u^\varepsilon), \qquad u^\varepsilon(\rho,\varphi,\tau)\big|_{\tau=0} = \widetilde{\psi}(\rho,\varphi),$$

where

$$\overline{L}_\varepsilon u^\varepsilon \equiv \varepsilon\tilde{a}(\rho,\varphi,\tau)\frac{\partial^2 u^\varepsilon}{\partial \rho^2} + \tilde{c}(\rho,\varphi,\tau)u^\varepsilon - \frac{\partial u^\varepsilon}{\partial \tau}, \qquad \sigma(\sqrt{\varepsilon},u^\varepsilon) = O(\sqrt{\varepsilon}).$$

Theorem 1. *For the solution u^ε of problem (1)–(2), in a neighborhood of every characteristic of discontinuity the representation*

$$u^\varepsilon = z_0^\varepsilon + z_1^\varepsilon + \dots + z_n^\varepsilon + O(\sqrt{\varepsilon^{n+1}}),$$

is valid, where z_k^ε, $k = 0,1,\dots,n$, are determined recursively:

$$\overline{L}_\varepsilon z_0^\varepsilon = 0, \qquad z_0^\varepsilon\big|_{\tau=0} = \widetilde{\psi}(\rho,\varphi), \qquad \overline{L}_\varepsilon z_k^\varepsilon = -\sigma(\sqrt{\varepsilon},z_{k-1}^\varepsilon), \qquad z_k^\varepsilon\big|_{\tau=0} = 0,$$

$$k = 1,2,\dots,n,$$

and z_0^ε *has the asymptotics in* ε *found in (2), while* $z_k^\varepsilon = O(\sqrt{\varepsilon}^{\,k})$.

2. We proceed to the study of the behavior, as $\varepsilon \to 0$, of the solution $u^\varepsilon(x,t)$ of equation (1), satisfying the conditions

$$u^\varepsilon(x,t)\big|_{t=0} = \Psi_1(x); \qquad u^\varepsilon(x,t)\big|_{x_1=0} = \Psi_2(x_2,\dots,x_n,t), \tag{3}$$

where $(x,t) \in D$ $(0 \le x_1 < \infty,\ 0 \le t \le T)$; $\Psi_1(x)$ and $\Psi_1(x_2,\dots,x_n,t)$ are bounded and have bounded derivatives with respect to all $x_i,\ i = 1,2,\dots,n$, up to order $2n$, and $\Psi_1(0,x_2,\dots,x_n) = \Psi_2(x_2,\dots,x_n,0)$, and $B_1(x,t) < 0$. Then the solution $v(x,t)$ of problem (1)–(3) for $\varepsilon = 0$ will be continuous, while its first derivatives on characteristics issuing from the $(n-1)$-dimensional plane $t = 0,\ x_1 = 0$ (characteristics of discontinuity) may have discontinuities of the first kind. In this case, for the derivatives of the solution u^ε in a neighborhood of such characteristics, one can observe the phenomenon of an inner parabolic boundary layer. We shall find the asymptotics in ε, in a neighborhood of every characteristic of discontinuity, for the solution u^ε and its first derivatives.

Under the change of x and t to y and τ, the plane $x_1 = 0$ passes into a certain surface Γ, which intersects $y_1 = 0$ in the $(n-1)$-dimensional plane $\tau = 0,\ y_1 = 0$, the domain D into the domain D', and condition (3) into the condition

$$u^\varepsilon(y,\tau)\big|_{\tau=0} = \psi_1(y); \qquad u^\varepsilon(y,\tau)\big|_{\Gamma} = \psi_2. \tag{3'}$$

In this case the role of the surface of discontinuity will be played by $\tau = 0,\ y_1 = 0$, and

$$\overline{L}_\varepsilon u^\varepsilon = \varepsilon a_{11}(y,\tau) u^\varepsilon_{y_1 y_1} + c(y,\tau) u^\varepsilon - u^\varepsilon_\tau.$$

If z_0^ε is the solution of $\overline{L}_\varepsilon z_0^\varepsilon = 0$ under the condition $(3')$ (where $y_2, y_3, \dots, y_n$ are regarded as parameters), and z_k^ε are the solutions of

$$L_\varepsilon z_k^\varepsilon = -\sigma(\sqrt{\varepsilon}, z_{k-1}^\varepsilon), \qquad k = 1,2,\dots,n,$$

under homogeneous conditions $(3')$, then

$$u^\varepsilon(y,\tau) = z_0^\varepsilon + z_1^\varepsilon + \dots + z_n^\varepsilon + O(\sqrt{\varepsilon}^{\,n+1}), \tag{4}$$

where $z_k^\varepsilon = O(\sqrt{\varepsilon}^{\,k}),\ k = 1,2,\dots,n.$

It remains to study the behavior of z_0^ε as $\varepsilon \to 0$, i.e., the behavior as $\varepsilon \to 0$ of the solution of problem $(1')$–$(3')$ for $n = 1$ (we shall denote the solution of this problem by $w^\varepsilon(y,\tau)$). We shall assume the coefficients of equation $(1')$ to be given for all y and $\tau > 0$, and therefore $v(y,\tau)$ may be regarded as defined for all y and $\tau > 0$. Denote by $w_0^\varepsilon(y,\tau)$ the solution of $(1')$, satis-

satisfying the condition $w_0^\varepsilon(y,\tau)\big|_{\tau=0} = v(y,\tau)\big|_{\tau=0}$; then, as $\varepsilon \to 0$, $w_0^\varepsilon(y,\tau)$ and $w^\varepsilon(y,\tau)$ converge in the domain D' to the same limiting solution $v(y,\tau)$, and it can be shown that $w_0^\varepsilon(y,\tau) - w^\varepsilon(y,\tau) = O(\varepsilon)$ for $(y,\tau) \in D'$. Define w_k^ε, $k = 1, 2, \dots, n$, as solutions of equation $(1')$ under the conditions $w_k^\varepsilon\big|_{\tau=0} = v_k\big|_{\tau=0}$, where $v_0(y,\tau) = v(y,\tau)$, while $v_k(y,\tau)$, $k = 1, 2, \dots, n$, are solutions of $(1')$ for $\varepsilon = 0$, satisfying respectively the conditions $v_k(y,\tau)\big|_{\substack{\tau=0\\ y_1\geq 0}} = 0$, $v_k(y,\tau)\big|_\Gamma = (v_{k-1} - w_{k-1}^\varepsilon)\big|_\Gamma$; then $w_k^\varepsilon(y,\tau) = O(\varepsilon^k)$, $k = 1, 2, \dots, n$.

Theorem 2. For the solution $w^\varepsilon(y,\tau)$ and its derivative $\partial w^\varepsilon(y,\tau)/\partial y$ in a neighborhood of $y = 0$, the following representations hold

$$w^\varepsilon(y,\tau) = w_0^\varepsilon(y,\tau) + w_1^\varepsilon(y,\tau) + \dots + w_n^\varepsilon(y,\tau) + O(\varepsilon^{n+1}) \quad \text{for } \tau \in [0, T_1],$$

$$\frac{\partial w^\varepsilon(y,\tau)}{\partial y} = \frac{\partial w_0(y,\tau)}{\partial y} + \frac{\partial w_1^\varepsilon(y,\tau)}{\partial y} + \dots + \frac{\partial w_n^\varepsilon(y,\tau)}{\partial y} + O\left(\varepsilon^n \sqrt{\frac{\varepsilon}{\tau}}\right)$$

$$\text{for } 0 < \tau \leq T_1,$$

where the solutions $w_k^\varepsilon(y,\tau) = O(\varepsilon^k)$ were found above, and moreover

$$\frac{\partial w_k^\varepsilon(y,\tau)}{\partial y} = O\left(\frac{\varepsilon^k}{\sqrt{\varepsilon\tau}}\right), \quad k = 1, 2, \dots, n,$$

while $\partial w_0/\partial y$ has the asymptotics in ε found in (2).

In the general case, when $n > 1$, the following theorem is valid.

Theorem 3. For the solution u^ε of problem $(1')$–$(3')$ (and hence also of problem (1)–(3)), in a neighborhood of every characteristic of discontinuity there is a representation of the form (4), and $z_0^\varepsilon(y,\tau)$ satisfies Theorem 2.

3. Analogous results, with some modifications, can be obtained in studying the behavior, as $\varepsilon \to 0$, of the solution $u^\varepsilon(x,t)$ of equation (1) satisfying the conditions

$$u^\varepsilon(x,t)\big|_{t=0} = \Psi_1(x); \qquad u^\varepsilon(x,t)\big|_S = \Psi_2, \quad x \in \Omega, \quad t \in [0,T], \tag{5}$$

where Ω is some domain in the space $x_1, \dots, x_n$; S is its boundary. For simplicity we restrict ourselves to the case $n = 1$; then (5) takes the form

$$u^\varepsilon(x,t)\big|_{t=0} = \Psi_1(x); \qquad u^\varepsilon(x,t)\big|_{x=0} = \Psi_2(t); \tag{6}$$

$$u^\varepsilon(x,t)\big|_{x=1} = \Psi_3(t), \tag{7}$$

where $x \in [0,1]$, $t \in [0,T]$; $\Psi_1(x)$ and $\Psi_2(t)$ are n times continuously differentiable and bounded together with all their derivatives; $\Psi_3(t)$ is continuous and bounded; $\Psi_3(0) = \Psi_1(1)$; $\Psi_1(0) = \Psi_2(0) = 0$; the coefficient $B_1(x,t) < 0$. Let $v(x,t)$ be the solution of (1)–(6) for $\varepsilon = 0$. Then, along the characteristic $l(0,0)$, the first derivatives of the solution $v(x,t)$ may have a discontinuity of the first kind. In this case, for the derivatives of the solution $u^\varepsilon(x,t)$ in a neighborhood of $l(0,0)$, the phenomenon of an internal parabolic boundary layer is observed. Moreover, since $v(x,t)$ does not satisfy (7), in a neighborhood of $x = 1$ for the solution $u^\varepsilon(x,t)$ there is also another phenomenon—the phenomenon of an external boundary layer (^{3}).

Let us find the asymptotics in ε for the solution $u^\varepsilon(x,t)$ and its first derivatives in a neighborhood of $l(0,0)$. We shall assume $\Psi_1(x)$ and $\Psi_2(t)$ to be given, respectively, for all $x \geq 0$ and $t \geq 0$. Denote by $\bar{u}^\varepsilon(x,t)$ the solution of (1)–(6) for $x \geq 0$, $t \in [0,T]$; then, as $\varepsilon \to 0$, $\bar{u}^\varepsilon(x,t)$ and $u^\varepsilon(x,t)$ converge to $v(x,t)$ for $0 \leq x < 1$. It turns out that one can construct a function

$w^\varepsilon(x,t)$, which cancels the discrepancy in the boundary conditions at $x = 1$ between $\overline{u}^\varepsilon(x,t)$ and $u^\varepsilon(x,t)$ (i.e., $(\overline{u}^\varepsilon(x,t) - u^\varepsilon(x,t))|_{x=1}$), and

$$u^\varepsilon(x,t) = \overline{u}^\varepsilon(x,t) + w^\varepsilon(x,t) + O(\varepsilon^{n+1})$$

for $x \in [0,1]$, $t \in [0,T]$. The function $w^\varepsilon(x,t)$ has the form of an external boundary layer and is constructed by the method developed by L. A. Lyusternik and M. I. Vishik[3]. In this case the corresponding asymptotics for the first derivatives of the solution $u^\varepsilon(x,t)$ in a neighborhood of $l(0,0)$, outside certain neighborhoods of the origin and of the point of intersection of $l(0,0)$ with $x = 1$, is obtained by differentiating only $\overline{u}^\varepsilon(x,t)$.

In the case $n > 1$, analogous arguments are carried out.

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Note: Figure translations are in progress. See original paper for figures.

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