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Abstract

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ON A BOUNDARY-VALUE PROBLEM IN THE THEORY OF FUNCTIONS

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MATHEMATICS

1. Let G be an $(m+1)$-connected domain of the plane of the variable $z = x + iy$, bounded by the contour $\Gamma = \Gamma_0 + \Gamma_1 + \dots + \Gamma_m$, consisting of closed nonintersecting smooth curves Γ_j, $j = 0, 1, \dots, m$, of which Γ_0 contains all the others inside it. By $\mathfrak{A}$ we shall denote the class of functions holomorphic inside G and continuous in $G + \Gamma$.

In the present work the following boundary-value problem is considered:

Problem I. Find all pairs of functions (φ, ψ), $\varphi \in \mathfrak{A}$, $\psi \in \mathfrak{A}$, satisfying the boundary condition

$$\varphi + \chi\overline{\psi} = f \quad \text{on } \Gamma, \tag{1}$$

where f and χ are given complex functions of the point $t \in \Gamma$, Hölder-continuous on Γ. The bar denotes passage to the complex-conjugate quantity. We assume that $\chi \neq 0$ on Γ. For simplicity we shall take $|\chi| \equiv 1$. Otherwise, in all subsequent formulations one should replace $\overline{\chi}$ by $1/\chi$. For the case of a simply connected domain, Problem I is easily reduced, by means of a conformal mapping of the domain G onto the unit disk, to the well-studied Hilbert boundary-value problem in the class of analytic functions. Also for simply connected domains, in the paper ([1]), a generalization of the Gazeman-type problem for Problem I is studied. For the applications indicated in item 4, the possibility of a complete solution of Problem I for multiply connected domains is essential. Below, the investigation of Problem I is carried out in the same spirit as the investigation of the Riemann-Hilbert boundary-value problem proposed by I. N. Vekua ([2]). This means, in particular, that for the study of Problem I a homogeneous adjoint problem is brought in, which is likewise a problem of type I.

The main result of the paper is the establishment of simple criteria for the solvability of Problem I, the study of its relation to the adjoint problem, and the computation of the number of linearly independent solutions of the homogeneous problem I_0 $(f \equiv 0)$.

Along with Problem I we shall consider the following homogeneous boundary-value problem, which we shall call the problem adjoint to Problem I, and denote by I^*.

Problem I_0^***.** Find all pairs of functions (φ, ψ), $\varphi \in \mathfrak{A}$, $\psi \in \mathfrak{A}$, satisfying the boundary condition

$$\varphi + \chi \overline{t'^2 \psi} = 0 \quad \text{on } \Gamma, \tag{2}$$

where $t = t(s)$ is the equation of the contour Γ; s is arc length.

It is not difficult to see that the problem adjoint to I_0^* coincides with the original homogeneous problem.

The number $n = \dfrac{1}{2\pi} \Delta_\Gamma \arg \chi$ will be called the **index** of problem I. If n^* is the index of problem I_0^*, then, obviously, $n^* = -n + 2(m-1)$. Solutions (φ, ψ) and (φ_1, ψ_1) of problem I_0 will be called **linearly independent** if the pairs of functions (φ, ψ) and (φ_1, ψ_1) are linearly independent over the field of real numbers. By l and l^* we shall denote the number of linearly independent solutions of problems I_0 and I_0^*, respectively.

2. Below we give, without proof, the main results of the investigation of problem I.

Theorem 1. *Every solution of problem I in the class $\mathfrak{A}$ is continuous in the sense of Hölder in the closed domain $G + \Gamma$.*

Theorem 1 makes it possible to apply the classical theory of singular integral equations to the investigation of problem I.

By N_φ^G (N_ψ^G) we shall denote the number of zeros of the function φ (ψ) inside the domain G, counted with their multiplicities. Let N_φ^k (N_ψ^k) be the number of zeros of φ (ψ), counted with multiplicity, on the contour Γ_k. Put $N_\varphi^\Gamma = \sum_{k=0}^m N_\varphi^k$, $\quad N_\psi^\Gamma = \sum_{k=0}^m N_\varphi^k$.

Theorem 2. *For any solution of problem I_0, the numbers N_φ^k and N_ψ^k are determined, the numbers N_φ^G, N_ψ^G, N_φ^Γ, and N_ψ^Γ are finite, and moreover*

$$2(N_\varphi^G + N_\psi^G) + N_\varphi^\Gamma + N_\psi^\Gamma = 2n.$$

Theorem 2 can be established by passing to the canonical forms of problem I_0 and applying the principle of the argument.

From Theorem 2 we immediately obtain:

Corollary. *If $n < 0$, the homogeneous problem I_0 has no solution different from zero.*

Theorems 3 and 4 establish the relation of problem I with the conjugate homogeneous problem I_0^*.

Theorem 3. *For the solvability of the nonhomogeneous problem I it is necessary and sufficient that integral equalities of the form*

$$\int_\Gamma \varphi f\,dt = 0, \qquad j = 1, 2, \dots, l^*, \tag{3}$$

hold, where φ_j, $j = 1, \dots, l^*$, is a complete system of solutions of the conjugate homogeneous problem. If $l^* = 0$, then the nonhomogeneous problem I is solvable for any right-hand side.

Theorem 4. *The difference between the numbers of linearly independent solutions of the problems I^0 and I_0^* is equal to the difference of the indices of these problems*

$$l - l^* = n - n^* = 2n - 2(m-1). \tag{4}$$

Theorems 3 and 4 are established on the basis of the theory of singular integral equations. Namely, on the basis of Theorem 1, the desired pair φ and ψ is represented in the form

$$\varphi(z) = \frac{1}{\pi i}\int_\Gamma \frac{\mu(t)\,dt}{t-z} + iC, \qquad \psi(z) = \frac{1}{\pi i}\int_\Gamma \frac{\nu(t)\,dt}{t-z} + iC_1 \tag{5}$$

of Cauchy-type integrals with real densities ([13]), for which from the boundary condition (1) we obtain a system of singular integral equations of normal type, equivalent to problem I. The solutions of the adjoint system of integral equations turn out to be connected by simple formulas with the boundary values of the solutions of the conjugate homogeneous problem.

Let us note that the totality of conditions of the form (3) is equivalent to the totality of conditions $\mathrm{Im}\int_\Gamma \varphi_j f\,dt = 0$, $j = 1, 2, \dots, l^*$. This follows from the fact that, as is easily seen, together with the pair (φ, ψ), the pair $(i\varphi, -i\psi)$ is a solution of the homogeneous problem I_0^*. In particular, it follows from this that the numbers l and l^* are always even.

On the basis of Theorems 3 and 4 and of the corollary to Theorem 2, we obtain:

Theorem 5. *If $n > 2(m-1)$, then the nonhomogeneous problem I is always solvable, and the homogeneous problem I_0 has exactly $2n - 2(m-1)$ linearly independent solutions. If $n < 0$, then the nonhomogeneous problem is solvable only when $l^* = 2(m-1) - 2n$ conditions of the form (3) are fulfilled. In this case the solution of the problem is unique.*

Thus the number l cannot be computed only in the "special" case of problem I, when its index $0 \le n \le 2(m-1)$. Generally speaking, in the special case the formula following from Theorem 5, $l = l_n = \max(0, 2n - 2(m-1))$, for $n < 0$ or $n > 2(m-1)$, may turn out to be false. For example, one can show that for $n = 0$ one may have either $l = 0$ or $l = 2$, and examples realizing both possibilities can be indicated. Hence, and from (4), we obtain for $l_{2(m-1)}$ two possible values: $l_{2(m-1)}$ is equal to $2m - 2$ or $2m$. This means that, whereas for $n < 0$ or

$n > 2m-2$ the number l_n is determined only by the topological properties of the data of the problem—the numbers n and m—in the "special" case it may also depend on other data of the problem, on the configuration of the contour Γ, and on special properties of the coefficient χ. Without dwelling further on this question, let us note only that the "special" case of the problem under consideration is analogous to the special case of the Riemann-Hilbert problem [(4)], and the qualitative pictures of the solvability relations of both problems are completely analogous. The number of linearly independent solutions of the problem I_0 in the "special" case is estimated from above in the following theorem.

Theorem 6. *For $0 \leq n \leq 2m-2$, the number of linearly independent solutions of the problem I_0 satisfies the inequality*

$$l_n \leq n+2.$$

For $n=0$ and $n=2m-2$ this estimate is attained.

The lower estimate for the number l_n is obtained from Theorem 4. Obviously, always $l_n \geq \max(0, 2n-2(m-1))$.

Remark 1. It is not difficult to show that, by means of a series of substitutions using, in particular, the solution of the modified Dirichlet problem [(3)], the homogeneous problem I_0 can be reduced to the canonical form

$$\varphi + t^n e^{c(t)} \overline{\psi} = 0 \quad \text{on } \Gamma,$$

where $c(t) = c_k' + ic_k'' = \text{const}$ on Γ_k is a piecewise-constant complex function defined on the contour Γ. The canonical form of the problem I_0 is used for the proof of Theorems 1 and 6.

Remark 2. Let φ and ψ be solutions of the problem I_0. Then the product $w = \varphi \cdot \psi$ satisfies the homogeneous Riemann-Hilbert problem $\mathrm{Im}(w\overline{\chi}) = 0$ on Γ, and moreover $w\overline{\chi} = \rho(t),\ t \in \Gamma,\ \rho(t) \leq 0$ everywhere on Γ. In particular, solutions of the problem I_0 may be used to construct solutions of the homogeneous Riemann-Hilbert problem satisfying the indicated additional condition.

3. Problem I can also be posed for the class of generalized analytic functions in the domain G, i.e. the class $\mathfrak{A}(A,B,G)$ [(5)]. The methods developed by I. N. Vekua [(2)] make it possible, on the basis of the study of problem I in the class $\mathfrak{A}$, to establish analogous results in the class $\mathfrak{A}(A,B,G)$. Naturally, in this case the solutions of the conjugate problem are sought in the class

$\mathfrak{A}^*(A,B,G) = \mathfrak{A}(-A,-\overline{B},G)$. The conditions (3) then take the form $\mathrm{Im}\int_\Gamma \varphi_j f\,dt = 0$. In proving Theorems 3 and 4 one should then use representations of functions of the class $\mathfrak{A}(A,B,G)$ in the form of generalized Cauchy-type integrals with real densities:

$$\varphi(z) = \frac{1}{\pi i}\int_\Gamma \Omega_1(z,t)\mu(t)\,dt + \Omega_2(z,t)\mu(t)\,\overline{dt}.$$

Such representations are given in the work ([5]).

With the indicated changes, Theorems 1, 2, 3, 4, and 5, as well as the corollary from Theorem 2, in their literal formulation are valid for Problem I in the class of generalized analytic functions.

4. Let a be some complex function given on Γ, $n = \dfrac{1}{2\pi}\Delta_\Gamma \arg a,\ n > m-1$. Problem I in the class $\mathfrak{A}$ arises naturally in attempting to construct in the domain G an operator of the form

$$T(\omega) = -\frac{1}{\pi}\iint_G \left[T_1(z,t)\omega(t) + T_2(z,t)\overline{\omega(t)}\right] dG_t, \tag{6}$$

defined on the class $L_p(G)$ of functions summable with degree $p > 2$ in the domain G and satisfying the conditions

$$\operatorname{Re}[aT(\omega)] = 0 \quad \text{on } \Gamma; \qquad \frac{\partial T(\omega)}{\partial \bar{z}} = \omega \quad \text{in } G \tag{7}$$

for all $\omega \in L_p(G)$. It is not difficult to verify that, if

$$T_1(z,t) = \frac{1}{t-z} + \varphi(z,t), \qquad T_2(z,t) = \psi(z,t),$$

where $\varphi(z,t)$ and $\psi(z,t)$ are solutions of Problem I for $\chi = \dfrac{a}{\bar{a}},\ f = -\dfrac{1}{t-z}$, then the operator of the form (6) will satisfy the conditions (7). An operator of the form (6) is analogous to the integral operator whose kernel is the Green function for the simplest boundary-value problems. In particular, it can be applied in the study of the Riemann-Hilbert problem for a general elliptic system of first order and in the study of the problem of an "oblique" derivative for a second-order equation in multiply connected domains.

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Note: Figure translations are in progress. See original paper for figures.