MATHEMATICS

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Abstract

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SOME CRITERIA FOR THE ABSENCE OF CLOSED TRAJECTORIES

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1°. Below a number of sufficient criteria are established for the absence, in a certain domain of the plane, of closed trajectories of a dynamical system described by the given differential system with continuous and twice continuously differentiable right-hand sides. The indicated criteria are not contained in Bendixson' s criterion ((1), p. 142). Nor do they follow from Poincaré' s criterion ((1), p. 144), if in the latter the topological system is assumed fixed.

2°. **Basic notation.** The symbol $\equiv$ should be understood as "is denoted below by" ; $E^2 \equiv$ the two-dimensional real Euclidean space; $X \equiv$ an open subset of E^2; $x \equiv (x_1, x_2) \in X$; $\|x\| \equiv (x_1^2+x_2^2)^{1/2}$; $p(x) \equiv$ a real vector-function $(p_1(x), p_2(x))$, defined and twice differentiable for all $x \in X$; $D \equiv (dx/dt = p(x),\ x \in X,\ t \in (-\infty, \infty))$; $X_D \equiv$ the set of singular points of D; $V \equiv$ a bounded subset of X; $h \equiv$ the diameter of V; $\underline{h} \equiv$ the exact upper bound of the diameters of circles inscribed in V; $G \equiv$ a subset of X; $G' \equiv G \setminus X_D$;

$$k(x) \equiv p_1^2(x)\,\partial p_2(x)/\partial x_1 + p_1(x)p_2(x)(\partial p_2(x)/\partial x_2 - \partial p_1(x)/\partial x_1)-$$

$$- p_2^2(x)\,\partial p_1(x)/\partial x_2;$$

$$k(x;t) \equiv |k(x)| - t\|p(x)\|^3;$$

$$m(x) \equiv p(x)\,\partial\big(k(x)/\|p(x)\|^3\big)/\partial x \quad \text{for } x \in G';$$

$$\Delta(x) = [0, \|p(x)\|^2/k(x)] \quad \text{for } k(x) \neq 0;$$

$$\Delta(x) = (-\infty, \infty) \quad \text{for } k(x) = 0;$$

$$\xi(G) \equiv \bigcup_{x\in G'} \bigcup_{t\in\Delta(x)} (x+t)(-p_2(x),\, p_1(x));$$

$$\eta(G) \equiv \bigcup_{x\in G'} \bigcup_{t\in(-\infty,\infty)} (x+t(-p_2(x),\, p_1(x))).$$

3°. **Definitions.** $\gamma \equiv$ a closed Jordan curve; $I_\gamma \equiv$ the interior of γ (γ encloses V) $\equiv$ ($V \subset I_\gamma$).

(G is acyclic (D)) $\equiv$ (there exists no closed trajectory of D having common points with G, in particular ($G \cap X_D = 0$)); (G is acyclic (D) in itself) $\equiv$ (there exists no closed trajectory of D situated entirely in G); (G is acyclic (D) with respect to V) $\equiv$ (there exists no closed trajectory of D situated entirely in G and enclosing V).

4°. **Propositions** $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$. $\mathfrak{A} \equiv$ (if $k(x, 2/\underline{h}) > 0$ for all $x \in G'$, then G is acyclic (D) with respect to V); $\mathfrak{B} \equiv$ (if $k(x) \neq 0$ for all $x \in G'$ and $\xi(G) \cap V \neq V$, then G is acyclic (D) with respect to V); $\mathfrak{C} \equiv$ (if $\eta(G) \cap V \neq V$, then G is acyclic (D) with respect to V); $\mathfrak{D} \equiv$ (if $k(x, 2/h) < 0$ for all $x \in V$, then V is acyclic (D) in itself); $\mathfrak{E} \equiv$ (if $m(x) \neq 0$ for all $x \in G'$, then G' is acyclic (D) in itself).

5°. **Propositions** $\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{C}_1, \mathfrak{D}_1, \mathfrak{E}_1$. The propositions 4° are not difficult to translate into geometric language. Thus there arise the propositions (respectively) $\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{C}_1, \mathfrak{D}_1, \mathfrak{E}_1$. $\mathfrak{A}_1 \equiv$ (if at every point of G' the curvature of the trajectory D is greater than $2/h$, then G is acyclic (L') with respect to V); $\mathfrak{B}_1 \equiv$ (if the trajectories D have no points of straightening in G', and if the set $\xi(G)$, formed by the segments joining the points of G' with the centers of curvature of the trajectories D passing through these points, does not cover V, then G is acyclic (D) with respect to V); $\mathfrak{C}_1 \equiv$ (if the set $\eta(G)$, formed by the normals to the trajectories D drawn at all points of G', does not cover V, then G is acyclic (D) with respect to V); $\mathfrak{D}_1 \equiv$ (if $V \cap X_D = 0$ and the curvature of the trajectory D passing through any point of V is less than $2/h$, then V is acyclic (D) in itself); $\mathfrak{E}_1 \equiv$ (if none of the trajectories D passing through G' has stationary points of curvature in G', then G' is acyclic (D) in itself).

6°. **Propositions** $\mathfrak{A}_2, \mathfrak{B}_2, \mathfrak{C}_2, \mathfrak{D}_2, \mathfrak{E}_2$. The propositions $\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{C}_1, \mathfrak{D}_1, \mathfrak{E}_1$, and consequently also the propositions $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}$, follow (respectively) from the following geometric facts ($l_1 \equiv$ (a closed trajectory of the system D); $l_2 \equiv$ (l_1, enclosing V); $\varkappa l_i x =$ (the curvature of l_i at the point $x \in l_i$), $i = 1, 2$); $\mathfrak{A}_2 \equiv$ (if $G' \supset l_2$, then $\varkappa l_2 x \leq 2/h$ at least at one point $x \in G'$); $\mathfrak{B}_2 \equiv$ (if $G' \supset l_2$ and l_2 has no points of straightening, then every point of V lies on the segment joining some point of $l_2 \cap G'$ with the corresponding point of the evolute of l_2); $\mathfrak{C}_2 \equiv$ (if $G' \supset l_2$, then every point of V lies on some normal to l_2); $\mathfrak{D}_1 \equiv$ (if $V \cap X_D = 0$ and $V \supset l_1$, then $\varkappa l_1 x \geq 2/h$ for some $x \in V$); $\mathfrak{E}_2 \equiv$ (if $G' \supset l_1$, then $\varkappa l_1 x$ attains an extremum for some $x \in G'$).

7°. **Remarks.** 1) From the propositions 4° it is not difficult to derive criteria for acyclicity (D) of a domain G, if, for example, one assumes that G is bounded by curves each of which consists either only of entry points or only of exit points. The latter will certainly be fulfilled if the boundary of G consists of smooth closed curves that are curves without contact. 2) The twice differentiability of $p(x)$ is used only in the propositions $\mathfrak{C}, \mathfrak{C}_1, \mathfrak{C}_2$; in the remaining propositions continuous differentiability of $p(x)$ is sufficient. 3) If at some points $x \in G'$ $k(x) = 0$, then the condition $\xi(G) \cap V \neq V$ is sufficient for the absence in G' of convex trajectories D enclosing V (cf. $\mathfrak{B}$).

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REFERENCES

1. V. V. **Nemytskii**, V. V. **Stepanov**, *Qualitative Theory of Differential Equations*, 2nd ed., Moscow-Leningrad, 1949.

Note: Figure translations are in progress. See original paper for figures.