A. I. PEROV

In the present article new uniqueness theorems are proposed for differential equations

1958

SovietRxiv

Abstract

Full Text

A. I. PEROV

ON UNIQUENESS THEOREMS FOR ORDINARY DIFFERENTIAL EQUATIONS

(Presented by Academician A. N. Kolmogorov, 29 I 1958)

In the present article new uniqueness theorems are proposed for differential equations

$$\frac{dx}{dt} = f(t, x), \tag{1}$$

which strengthen a number of previously known results.

Equation (1) is considered in a real Banach space E. The derivative is understood in the strong sense. The operator $f(t, x)$, with values in E, is assumed to be defined for $0 < t \leq \alpha$, $\|x - x_0\| \leq \beta$. We study continuous solutions satisfying equation (1) for $0 < t \leq \alpha$ and satisfying the initial condition

$$x(0) = x_0, \tag{2}$$

In the case when E is an n-dimensional space, the assertions established below become uniqueness theorems for systems of ordinary differential equations. If E is a space of sequences, then equation (1) is an infinite system of ordinary differential equations. Finally, if E is a space of functions, then equation (1) is an equation with partial derivatives (for example, an integro-differential equation).

The question of the existence of solutions of problem (1)–(2) is not considered by us. For the case in which $f(t, x)$ is an operator continuous jointly in the variables, see the existence theorems in ([1]). The case in which $f(t, x)$ is continuous jointly in the variables only for $t > 0$ was investigated by V. A. Chechik ([2]) for finite systems of ordinary differential equations. Without substantial changes, the Carathéodory existence theorems ([3,4]) carry over to equations in Banach spaces.

1. In what follows, by $\lambda(z)$ we denote a continuous functional defined on the ball $\|z\| \leq 2\beta$, for which $\lambda(0) = 0$, $\lambda(z) > 0$ when $\|z\| > 0$. It is assumed that the increment of the functional $\lambda(z)$ can be estimated in the following way:

$$\lambda(z + h) - \lambda(z) \leq D(z, h) + \alpha(z, h), \tag{3}$$

where the functional $D(z, h)$ is continuous in h and semi-homogeneous:

$$tD(z,h) \leq D(z,th), \qquad t \geq 0, \tag{4}$$

and the functional $\alpha(z,h)$, for each fixed value of z, satisfies the condition

$$\lim_{h\to 0} \frac{\alpha(z,h)}{\|h\|} = 0. \tag{5}$$

As the functional $\lambda(z)$ there may occur a Fréchet-differentiable functional. In many cases it is convenient to consider various-

norms (even non-differentiable ones). In the case where E is one-dimensional, it is convenient to put $\lambda(z) = |z|$. Then

$$D(z,h) = \begin{cases} \operatorname{sign} z \cdot h, & z \neq 0, \\ |h|, & z = 0. \end{cases}$$

Let us give one more example. Let E be an n-dimensional space, $z = \{z_1, \dots, z_n\}$, $h = \{h_1, \dots, h_n\}$. If we put

$$\lambda(z) = \left(\sum_{i=1}^{n} |z_i|^p\right)^{1/p}, \qquad p \geqslant 1, \tag{6}$$

then as $D(z,h)$ one may take

$$D(z,h) = \begin{cases} \lambda(z)^{1-p} \sum_{i=1}^{n} |z_i|^{p-1}|h_i|, & z \neq 0, \\ \lambda(h), & z = 0. \end{cases}$$

2. Throughout the article it is assumed that the right-hand side of equation (1) satisfies the condition

$$D[x-y,\ f(t,x)-f(t,y)] \leqslant k\,\frac{\lambda(x-y)}{t} + t^k a(t) L\left[\frac{\lambda(x-y)}{t^k}\right] \tag{7}$$

for $0 < t \leqslant \alpha$, $\|x-x_0\| \leqslant \beta$, $\|y-x_0\| \leqslant \beta$, $x \neq y$, $\lambda(x-y) \leqslant \gamma t^k$. Here k is some nonnegative number. Concerning the function $a(t)$ it is assumed that

$$\lim_{t\to+0} \int_t^\alpha a(\tau)\,d\tau < +\infty.$$

Concerning the function $L(v)$ it is assumed that it is continuous for $0 \leqslant v \leqslant \gamma$, positive for $v > 0$, and

$$\int_0^\gamma \frac{dv}{L(v)} = +\infty.$$

We shall say that two solutions $x(t)$ and $y(t)$ of problem (1)–(2) are **equivalent** (belong to one equivalence class) if

$$\lim_{t \cdot 0} \frac{\|x(t) - y(t)\|}{t^k} = 0. \tag{8}$$

Lemma 1. *If condition (7) is fulfilled, then each class of equivalent solutions consists of no more than one element.*

3. In order to obtain from Lemma 1 the uniqueness of the solution of problem (1)–(2), it is necessary to impose on the right-hand side of equation (1) such restrictions under which any solutions are equivalent.

Suppose, for example, that the right-hand side of equation (1) is defined for $t = 0$, $x = x_0$, and that solutions satisfying the equation for $0 \leqslant t$ are considered. Then any two solutions will satisfy condition (8) for $0 \leqslant k < 1$. In this case the uniqueness theorem will hold if in condition (7) $0 \leqslant k < 1$.

Putting $k = 0$, we arrive at the Osgood–Tamarkin theorem ([4,5]). More precisely, we obtain a generalization of the Osgood–Tamarkin theorem to the case of equations in Banach spaces.

Putting $k = 1$, we obtain a generalization of the Rosenblatt–Nagumo–Perron theorem (see ([4])). The generalization is obtained even for the case of a finite system

differential equations, since in order to obtain the Rosenblatt–Nagumo–Perron conditions it is necessary to put $a(t) \equiv 0$. Other generalizations of the indicated theorems to the case of Banach spaces may be found in ([6,7]).

By choosing different functions $a(t)$ and $I_\nu(v)$, one can obtain generalizations, to equations in Banach spaces, of other known uniqueness theorems (see ([4])). Let us also note that the use of the functional (6) leads to a theorem close to Cviher' s uniqueness conditions ([8]).

In the case when equation (1) is singular in the sense of V. A. Chechik, i.e. the solution satisfies the equation only for $0 < t \leq \alpha$, condition (8) is fulfilled without additional assumptions only in the case $k = 0$. Let us note that in this case ($k = 0$) Lemma 1 implies the uniqueness theorem proved for a more particular form of equations by V. A. Chechik ([2]).

For the case of a singular equation, for $0 \leq k \leq 1$ we shall additionally suppose that

$$D[x - y,\ f(t, x) - f(t, y)] \leq N(t, \lambda(x - y)) \tag{9}$$

for $0 < t < \alpha$, $\|x - x_0\| \leq \beta$, $\|y - x_0\| \leq \beta$, $x \neq y$. With respect to the function $N(t, u)$ we shall suppose that it is continuous for $0 \leq t \leq \alpha$, $u \geq 0$, and that $N(t, 0) \equiv 0$.

Lemma 2. *If condition (9) is fulfilled, then any two solutions of problem (1)–(2) satisfy condition (8) for* $0 \leq k \leq 1$.

4. In the subsequent arguments one more type of restriction on the right-hand side of equation (1) is used. These restrictions have the form

$$D[x - y,\, f(t, x) - f(t, y)] \leq b(t) M(\lambda(x - y)) \tag{10}$$

for $0 < t < \alpha$, $\|x - x_0\| \leq \beta$, $\|y - x_0\| \leq \beta$, $x \neq y$, $\lambda(x - y) \leq \delta$. Here $b(t)$ is continuous and positive for $0 \leq t \leq \alpha$, the function $M(u)$ ($M(0) = 0$) is continuous for $0 \leq u \leq \delta$ and positive for $0 < u \leq \delta$, and moreover

$$\int_0^\delta \frac{du}{M(u)} < +\infty.$$

It is also assumed that, for sufficiently small t, the inequality

$$\int_0^t b(\tau)\, d\tau \leq \int_0^{\varepsilon t^k} \frac{du}{M(u)}, \tag{11}$$

is fulfilled, whatever $\varepsilon > 0$ may be.

Lemma 3. *If condition (10) is fulfilled, then any two solutions of problem (1)–(2) satisfy condition (8), in which k is the number from (11).*

5. As we have already mentioned, combining Lemma 1 with the uniqueness conditions for a class of equivalent solutions gives uniqueness theorems for solutions.

Theorem 1. *Suppose condition (7) is fulfilled, in which* $0 \leq k \leq 1$. *Then the solution of problem (1)–(2) satisfying equation (1) also at* $t = 0$ *is unique.*

Suppose condition (7) is fulfilled, in which $k > 1$. *Then, for uniqueness of the solution of problem (1)–(2) satisfying equation (1) also at* $t = 0$, *it is sufficient that condition (10) be fulfilled.*

Theorem 2. *Suppose condition (7) is fulfilled. Then, for uniqueness of the solution of problem (1)–(2) satisfying equation (1) for* $t > 0$, *it is sufficient that, for* $0 < k \leq 1$, *condition (9) be fulfilled, and for* $k > 1$*—condition (10).*

6. If, in the hypotheses of Theorem 1, for $k > 1$ one sets $\lambda(z) = \|z\|$,

$$D(z, h) = \|h\|, \quad a(t) \equiv 0, \quad b(t) \equiv 1, \quad M(u) = p u^\alpha, \quad \alpha > 1 - \frac{1}{k},$$

then we arrive at the theorem of Krasnosel' skii–Krein ([7]).

The author expresses his gratitude to M. A. Krasnosel' skii for his attention and advice.

Voronezh State
University

Received
29 I 1958

REFERENCES

[1] M. A. Krasnosel' skii, S. G. Krein, *Tr. seminara po funktsional' nomu analizu*, Voronezh, vol. 2 (1956).
[2] V. A. Chechik, DAN, 108, No. 5 (1956).
[3] C. Carathéodory, *Vorlesungen über reelle Funktionen*, Leipzig–Berlin, 1918.
[4] J. Sansone, *Ordinary Differential Equations*, 2, IL, 1954.
[5] I. G. Petrovskii, *Lectures on the Theory of Ordinary Differential Equations*, 1952.
[6] M. A. Krasnosel' skii, S. G. Krein, DAN, 102, No. 1 (1955).
[7] M. A. Krasnosel' skii, S. G. Krein, *Uspekhi Mat. Nauk*, 11, No. 1, 67 (1956).
[8] G. Zwirner, *Rend. Sem. Mat. di Roma* (4), 1 (1937).

Note: Figure translations are in progress. See original paper for figures.