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Abstract

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MATHEMATICS

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ON THE THEORY OF PERTURBATIONS OF THE CONTINUOUS SPECTRUM

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K. O. Friedrichs ([1,2]) investigated the character of the spectrum for the operator $L = L_0 + \varepsilon K$, where L_0 is the operator of multiplication by the independent variable in a Hilbert space of abstract functions, and K is an integral operator. Under the fulfillment of a number of conditions of smoothness and decay type, similar to conditions (R) and (K) (see below), and for sufficiently small ε, he proved the unitary equivalence of the operators L and L_0, and also investigated the behavior as $|t| \to \infty$ of the solution of the Schrödinger equation with the operator L. The main part of his investigation consists in the study of the integral equation

$$r(\lambda, \mu) = k(\lambda, \mu) + i\pi\varepsilon k(\lambda, \mu) r(\mu, \mu) + \varepsilon P \int \frac{k(\lambda, \sigma) r(\sigma, \mu)}{\mu - \sigma} \, d\sigma \tag{I}$$

(the index P means that the integral is understood in the sense of the principal value) for the kernel $r(\lambda, \mu)$ of the operator R, by means of which the operator U, effecting the unitary transformation of L into L_0, is constructed. Friedrichs needed the smallness of ε in order to prove the solvability of equation (I). In the present paper the solvability of equation (I) is proved for arbitrary ε (we shall henceforth put $\varepsilon = 1$).

1. **Basic concepts and notation.** Let A be a complex Hilbert space of elements $x, y, ...$ with scalar product xy and norm $|x| = (xx)^{1/2}$. The set of measurable functions $x(\lambda)$ of the real variable λ, varying in the interval I, with values in A, for which * $\int |x(\lambda)|^2 \, d\lambda < \infty$, is a Hilbert space (which we shall denote by $\mathcal{H}$) if the scalar product

$$(x(\lambda), y(\lambda)) = \int x(\lambda) y(\lambda) \, d\lambda.$$

is introduced in it. On all functions for which

$$\int \lambda^2 |x(\lambda)|^2 \, d\lambda < \infty,$$

there is defined the operator $L_0 x(\lambda) = \lambda x(\lambda)$, which is self-adjoint in $\mathcal{H}$. Let, further, $k(\lambda, \mu)$, for fixed λ and μ from I, be a bounded operator in A; by $|k(\lambda, \mu)|$ we shall denote its norm, and by $\overline{k(\lambda, \mu)}$ the adjoint operator. In what follows, the kernel $k(\lambda, \mu)$ will be subject to the following conditions:

$$\left|(1 + |\lambda|^\beta + |\mu|^\gamma) k(\lambda, \mu)\right| \leq c_1, \qquad 0 < \beta, \gamma < 1; \tag{R_1}$$

$$|k(\lambda, \mu) - k(\lambda', \mu)| \leq c_2 |\lambda - \lambda'|^\beta; \tag{R_2}$$

$$\left|(1 + |\lambda|^\beta)(k(\lambda, \mu) - k(\lambda, \mu'))\right| \leq c_3 |\mu - \mu'|^\gamma; \tag{R_3}$$

$$|k(\lambda, \mu) - k(\lambda', \mu) - k(\lambda, \mu') + k(\lambda', \mu')| \leq c_4 |\lambda - \lambda'|^\beta |\mu - \mu'|^\gamma; \tag{R_4}$$

$$k(\lambda, \mu) = 0, \quad \text{if } \mu \text{ is on the boundary of } I; \tag{R_5}$$

* All integrals in what follows are extended over the interval I.

$k(\lambda, \mu)$ is a completely continuous operator in A for all λ and μ from I,
(T)

$$k(\lambda, \mu) = \overline{k(\mu, \lambda)}. \tag{K}$$

We shall call the operator kernel $r(\lambda, \mu)$ a **kernel of class (R)** if it satisfies conditions (R), except for the boundedness of $|\mu^\gamma r(\lambda, \mu)|$.

Lemma 1. *If conditions (R_1) and (R_2) are satisfied, with $\beta > 1/2$, then on every function from the domain of definition of L_0 the operator*

$$Kx(\lambda) = \int k(\lambda, \mu) x(\mu) \, d\mu$$

is defined; and if condition (K) is also satisfied, then the operator $L = L_0 + K$ is self-adjoint in H.

On the basis of our investigation of equation (I), carried out following Friedrichs' method, we can prove the following assertion:

Theorem 1. *Suppose that conditions (R), (T), and (K) are satisfied, with $\beta > 1/2$. Then the operator L has continuous spectrum on the interval I*

and at most a finite number of eigenvalues of finite multiplicity, which may lie both inside and outside the interval I. The part of the operator L acting in the invariant subspace corresponding to the continuous spectrum is unitarily equivalent to the operator L_0; i.e., there exists an operator U possessing the following properties:

$$LU = UL_0; \qquad U^*U = 1; \qquad UU^* = 1 - P. \tag{U}$$

Here P is the orthogonal projector onto the proper subspace of the operator L, corresponding to the discrete spectrum.

In particular, the properties (U) are possessed by the operators

$$U^{(\pm)} = \lim_{t \to \pm\infty} e^{-iLt} e^{iL_0 t},$$

where these limits exist in the strong sense. The operators $U^{(\pm)}$ are related to one another by the formula

$$U^{(+)} = U^{(-)} S,$$

where S is a unitary operator commuting with L_0.

Let us note that the operator S has important significance in quantum mechanics –it is the so-called scattering operator, or S-matrix.

2. We now indicate the main stages of our investigation of equation (I). It is convenient to consider it in the space of functions $u(\lambda)$ with values in A, which satisfy a Hölder condition with some exponent α, $0 < \alpha < 1$. The set of functions for which

$$\|u\|_\alpha = \sup_\lambda |u(\lambda)| + \sup_{\lambda\lambda'} \frac{|u(\lambda) - u(\lambda')|}{|\lambda - \lambda'|^\alpha} + \sup_\lambda |\lambda^\alpha u(\lambda)| < \infty,$$

is a complete Banach space if $\|u\|_\alpha$ is taken as the norm. We shall denote it by B_α. Consider in B_α the operator

$$T_\omega u(\lambda) = i\pi k(\lambda, \omega) u(\omega) + P \int \frac{k(\lambda, \sigma) u(\sigma)}{\omega - \sigma} \, d\sigma. \tag{1}$$

Lemma 2. *Suppose that conditions (R) are satisfied. Then the operator T_ω is defined on every function from B_α and is bounded in the norm of B_β. Moreover,*

$$\|T_\omega u\|_\beta \leqslant c_\omega \|u\|_\alpha,$$

where $c_\omega \to 0$ as $|\omega| \to \infty$, if J is bounded, and

$$\|(T_\omega - T_{\omega'})u\|_\beta \leq c|\omega - \omega'|^\delta \|u\|_\alpha, \qquad \delta = \min(\gamma, \alpha).$$

Lemma 3. Suppose the conditions of Lemma 2 are satisfied with $\beta > \alpha$, and condition (T) is satisfied. Then the operator T_ω is completely continuous in B_α.

We now consider the structure of the eigenfunctionals of the operator T^*_ω. Let $T^*_\omega l_\omega = l_\omega$. This means that for every $u \in B_\alpha$ we have:

$$(l_\omega, u) = \left(l_\omega, i\pi k(\lambda, \omega)u(\omega) + P \int \frac{k(\lambda, \sigma)u(\sigma)}{\omega - \sigma}\, d\sigma \right).$$

If conditions (R_1) and (R_2) are satisfied, then for fixed μ and arbitrary $x \in A$, $k(\lambda, \mu)x$ is an element of B_β and, all the more, of B_α. But then the expression $(l_\omega, k(\lambda, \mu)x)$ is meaningful, and it defines a linear functional in A. By Riesz' theorem,

$$(l_\omega, k(\lambda, \omega)x) = \varphi_\omega(\mu)x. \tag{2}$$

This equality defines the function $\varphi_\omega(\mu)$ with values in A. If conditions (R_1) and (R_3) are satisfied, then $\varphi_\omega \in B_\gamma$ and

$$(l_\omega, u) = i\pi\varphi_\omega(\omega)u(\omega) + P \int \frac{\varphi_\omega(\sigma)u(\sigma)}{\omega - \sigma}\, d\sigma. \tag{3}$$

If here, as $u(\lambda)$, we take $k(\lambda, \mu)x$, then after simple transformations we obtain for $\varphi_\omega(\mu)$ the equation

$$\varphi_\omega(\mu) = -i\pi\overline{k(\omega, \mu)}\, \varphi_\omega(\omega) + P \int \frac{k(\sigma, \mu)\varphi_\omega(\sigma)}{\omega - \sigma}\, d\sigma.$$

If, in addition, condition (K) is satisfied, then the equation for $\varphi_\omega(\mu)$ can be rewritten in the following form:

$$\varphi_\omega(\mu) = T_\omega\varphi_\omega(\mu) - 2\pi i k(\mu, \omega)\varphi_\omega(\mu).$$

From this it is not hard to see that $\varphi_\omega(\omega) = 0$. Indeed,

$$(l_\omega, \varphi_\omega) = (l_\omega, T_\omega\varphi_\omega) - 2\pi i(l_\omega, k(\mu, \omega)\varphi_\omega(\omega)) = (l_\omega, \varphi_\omega) - 2\pi i|\varphi_\omega(\omega)|^2.$$

We obtain the following result:

Lemma 4. Suppose conditions (R), (T), and (K) are satisfied. Then the eigenfunctionals of the operator T^*_ω corresponding to the eigenvalue 1 are constructed by formula (3) by means of the function $\varphi_\omega(\mu) \in B_\gamma$, defined by equality (2), and moreover $\varphi_\omega(\omega) = 0$.

On the basis of Lemmas 2–4 the following theorem is proved:

Theorem 2. Under the fulfillment of conditions (R), (T), and (K), equation (I) is always solvable, and the solution $r(\lambda, \mu)$ belongs to the class (R) and is a completely continuous operator in A.

3. As an example, consider the differential operator $Mu = -\Delta u + q(x)u$ in the whole three-dimensional space E_3. This operator is unitarily equivalent to an operator of type L of the general theory, where the interval $I = (0, \infty)$, and the space A is the space of square-integrable functions on the unit sphere. The operator $k(\lambda, \mu)$, for fixed λ and μ, is an integral operator on the unit sphere with kernel

$$k(\lambda, \mu; \alpha, \beta) = \left(\frac{1}{2\pi}\right)^3 \frac{(\lambda\mu)^{1/4}}{2} \int_{E_3} q(x) e^{i(\sqrt{\lambda}\alpha - \sqrt{\mu}\beta, x)}\, dx;$$

here α and β are unit vectors.

When a number of conditions of the type of smoothness and decrease of the function $q(x)$ are satisfied for $k(\lambda, \mu)$, the conditions (R) and (T) are satisfied and, if $q(x)$ is real, then condition (K) is satisfied, so that the assertions of Theorems 1 and 2 hold. In particular, for the solution of the Schrödinger equation

$$i\frac{\partial}{\partial t} z(\lambda, \alpha; t) = \lambda z(\lambda, \alpha; t) + \int_0^\infty d\mu \int d\beta\, k(\lambda, \mu; \alpha, \beta)\, z(\mu, \beta; t)$$

there exist the limits $\lim_{t\to\pm\infty} e^{i\lambda t} z(\lambda, \alpha; t) = z_\pm(\lambda, \alpha)$ in the mean with respect to λ and α, if $z(\lambda, \alpha; 0)$ is orthogonal to the eigenfunctions of the discrete spectrum of the operator L, and $z_+(\lambda, \alpha) = S(\lambda) z_-(\lambda, \alpha)$. For fixed λ, $S(\lambda)$ is a unitary operator on the unit sphere. This is the so-called scattering operator of the operator M.

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Note: Figure translations are in progress. See original paper for figures.