

# Academician of the Academy of Sciences of the Armenian SSR M. M. Dzhrbashyan and A. E. Avetisyan

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**Abstract**

**Full Text**

**MATHEMATICS**

Academician of the Academy of Sciences of the Armenian SSR M. M. Dzhrbashyan and A. E. Avetisyan

## INTEGRAL REPRESENTATION OF CERTAIN CLASSES OF FUNCTIONS ANALYTIC IN AN ANGULAR DOMAIN

In paper <sup>(1)</sup> integral representations were obtained for entire functions of finite order and normal type, integrable in the square of the modulus over certain systems of rays of the complex plane. The theory of special integral transforms, constructed in paper <sup>(2)</sup>, made it possible to establish that these representations are parametric, analogously to the well-known Paley-Wiener theorem <sup>(3)</sup> on entire functions of exponential type belonging to  $L_2(-\infty, \infty)$ .

In the present note we give formulations of a number of new results on the integral representation of various classes of functions analytic in an angular domain; moreover, under known conditions these representations are also parametric.

The proofs of the theorems stated below are obtained by methods close to those developed in papers <sup>(1, 2)</sup>.

1°. Below, for given  $\theta$  ( $-\pi \leq \theta \leq \pi$ ) and  $\gamma > 0$ , in the plane  $z$  cut along the ray  $\arg z = \pm\pi$ , we shall consider that branch of the function  $(ze^{-i\theta})^\gamma$  which takes real and positive values on the ray  $\arg z = \theta$ .

For given  $\theta$  ( $-\pi < \theta < \pi$ ),  $\rho \geq \frac{1}{2}$ , and  $\sigma \geq 0$ , denote by  $D(\theta, \rho, \sigma)$  the simply connected domain containing the ray  $\arg z = \theta$  ( $|z| \geq \sigma^{1/\rho}$ ) and the set of points  $\operatorname{Re}(ze^{-i\theta})^\rho > \sigma$ . Further denote by  $D_\alpha(\rho, \sigma)$  the sum of all domains of the form  $D(\theta, \rho, \sigma)$  for  $|\theta| \leq \frac{\pi}{2}\alpha$  ( $0 < \alpha < 2$ ), and by  $L_\alpha(\sigma)$  denote its boundary.

**Theorem 1.** Let the function  $f(z)$  be holomorphic in the angle  $\Delta_\alpha$  ( $|\arg z| < \frac{\pi}{2}\alpha$ ),  $0 < \alpha < 2$ , and continuous in the closed angle  $\bar{\Delta}_\alpha$ , except possibly for the point at infinity. Suppose, in addition, that in the same angle  $f(z)$  has order of growth not greater than  $\rho_1 \geq 0$ , and type not greater than  $\sigma \geq 0$ . Then the function

$$g(z) = \rho(ze^{-i\theta})^{\mu\rho-1} z^{-1} \int_0^\infty f(te^{-i\theta}) e^{-t^\rho(ze^{-i\theta})^\rho} t^{\mu\rho-1} dt, \quad (1)$$

where  $\rho \geq \max\{\rho_1, (2-\alpha)^{-1}\}$ ,  $\frac{1}{2} < \mu < \frac{1}{2} + \frac{1}{\rho}$ ,  $|\theta| \leq \frac{\pi}{2}\alpha$ , is holomorphic in the whole domain  $D_\alpha(\rho, \sigma)$ , and for  $z \in \Delta_\alpha$  the integral representation holds

$$f(z) = \int_{L_\alpha(\sigma+\delta)} E_\rho(z\zeta; \mu) g(\zeta) d\zeta, \quad (2)$$

where  $\delta > 0$ ,

$$E_\rho(z; \mu) = \sum_{n=0}^{\infty} \{\Gamma(\mu + n\rho^{-1})\}^{-1} z^n$$

is an entire function of Mittag-Leffler type, and the integral in (2) converges absolutely.

From this follows the following result on approximation by entire functions:

**Corollary 1.** If the function  $f(z)$  satisfies the conditions of Theorem 1, then the entire functions of order  $\rho \geq \max\{\rho_1, (2 - \alpha)^{-1}\}$  and of type  $\leq A$ , defined by the formula

$$f_A(z) = \int_{L_\alpha(\sigma+\delta, A)} E_\rho(z\zeta; \mu) g(\zeta) d\zeta, \quad (3)$$

where, for a given  $A > \sigma + \delta$ ,  $L_\alpha(\sigma + \delta, A)$  denotes the piece of the curve  $L_\alpha(\sigma + \delta)$  lying in the disk  $|\zeta| \leq A^{1/\rho}$ , converge uniformly to  $f(z)$  in every closed part of  $\Delta_\alpha$ , as  $A \rightarrow \infty$ .

We note that from Theorem 1, in particular, when  $0 < \alpha < 1$ ,  $\rho = \rho_1 = 1$ , we obtain Macintyre's theorem<sup>(4)</sup>, obtained by him by a method for which these assumptions are essential.

2°. Let us assign to the class  $M_2(\alpha, \omega)$  ( $0 < \alpha < 2$ ,  $-1 < \omega < +1$ ) the functions holomorphic in the angle  $\Delta_\alpha$  ( $0 < \alpha < 2$ ) and satisfying the conditions:

- a) the function  $f(z)$  has finite boundary values almost everywhere

$$f(re^{\pm i\frac{\pi}{2}\alpha});$$

- b) the integrals exist and are uniformly bounded

$$\int_0^\infty |f(re^{i\varphi})|^2 r^\omega dr < M < +\infty, \quad |\varphi| \leq \frac{\pi}{2}\alpha, \quad (4)$$

where  $M$  does not depend on  $\varphi$ .

**Theorem 2.** 1) If  $f(z) \in M_2(\alpha, \omega)$ , then for any  $\rho \geq (2 - \alpha)^{-1}$  and

$$\mu = \frac{1 + \rho + \omega}{2\rho}$$

the following integral representation is valid:

$$f\left(r^{\frac{1}{\rho}}e^{i\varphi}\right)r^{\mu-1} = \frac{d}{dr} \left\{ r^{\mu} \int_0^{\infty} E_{\rho}\left(r^{\frac{1}{\rho}}e^{i\varphi}\tau^{\frac{1}{\rho}}e^{i\left(\alpha+\frac{1}{\rho}\right)\frac{\pi}{2}}; \mu+1\right)v_{(-)}(\tau)\tau^{\mu-1}d\tau \right\} \\ + \frac{d}{dr} \left\{ r^{\mu} \int_0^{\infty} E_{\rho}\left(r^{\frac{1}{\rho}}e^{i\varphi}\tau^{\frac{1}{\rho}}e^{-i\left(\alpha+\frac{1}{\rho}\right)\frac{\pi}{2}}; \mu+1\right)v_{(+)}(\tau)\tau^{\mu-1}d\tau \right\} \quad (5)$$

for all  $|\varphi| \leq \frac{\pi}{2}\alpha$ , where  $v_{(\mp)}(\tau) \in L_2(0, \infty)$ . Moreover, for  $|\varphi| = \frac{\pi}{2}\alpha$  equality (5) holds almost everywhere on  $(0, +\infty)$ , and for  $z \in \Delta_{\alpha}$  it can be written in the form

$$f(z) = \int_0^{\infty} E_{\rho}\left(ze^{i\left(\alpha+\frac{1}{\rho}\right)\frac{\pi}{2}}\tau^{\frac{1}{\rho}}; \mu\right)v_{(-)}(\tau)\tau^{\mu-1}d\tau \\ + \int_0^{\infty} E_{\rho}\left(ze^{-i\left(\alpha+\frac{1}{\rho}\right)\frac{\pi}{2}}\tau^{\frac{1}{\rho}}; \mu\right)v_{(+)}(\tau)\tau^{\mu-1}d\tau, \quad (6)$$

where the integrals on the right converge absolutely.

- 2) If the function  $f(z)$  is defined by formula (5), where  $v_{(\mp)}(\tau) \in L_2(0, +\infty)$ , and the parameters  $\rho, \alpha, \omega, \mu$  have the previous values, then it is holomorphic in  $\Delta_{\alpha}$  and belongs to the class  $M_2(\alpha, \omega)$ .

Hence, in particular, follows Corollary 2, which is the well-known Paley-Wiener theorem <sup>(3)</sup>, but with a somewhat different definition of the class of representable functions.

**Corollary 2.** Functions of the class  $M_2(1, 0)$  are represented in parametric form as

$$f(z) = \int_0^{\infty} e^{-\tau z}v(\tau)d\tau, \quad (7)$$

where  $v(\tau)$  is an arbitrary function of the class  $L_2(0, \infty)$ .

From Theorem 2 there follows the following result on approximation by entire functions:

**Theorem 3.** Let  $f(z) \in M_2(\alpha, \omega)$ , and let the entire functions  $f_{\sigma}(z)$  of order not exceeding  $\rho \geq (2 - \alpha)^{-1}$ , and of type not exceeding  $\sigma$ , be defined by the formula

$$f_{\sigma}(z) = \int_0^{\sigma} E_{\rho}\left\{ze^{i\left(\alpha+\frac{1}{\rho}\right)\frac{\pi}{2}}\frac{1}{\tau^{\rho}}; \mu\right\}v_{(-)}(\tau)\tau^{\mu-1}d\tau \\ + \int_0^{\sigma} E_{\rho}\left\{ze^{-i\left(\alpha+\frac{1}{\rho}\right)\frac{\pi}{2}}\frac{1}{\tau^{\rho}}; \mu\right\}v_{(+)}(\tau)\tau^{\mu-1}d\tau, \quad (8)$$

where almost everywhere on  $(0, +\infty)$

$$v_{(\mp)}(\tau) = \frac{e^{\mp i \frac{\pi}{2}(1-\mu)}}{2\pi\rho} \frac{d}{d\tau} \int_0^\infty \frac{e^{\mp i\tau t} - 1}{\mp it} f(t^{1/\rho} e^{\mp i \frac{\pi}{2}\alpha}) t^{\mu-1} dt. \quad (9)$$

Then the entire functions  $f_\sigma(z)$ , as  $\sigma \rightarrow +\infty$ , converge to  $f(z)$  in the mean on every ray in the sense

$$\lim_{\sigma \rightarrow +\infty} \int_0^\infty |f(re^{i\varphi}) - f_\sigma(re^{i\varphi})|^2 r^\omega dr = 0, \quad |\varphi| \leq \frac{\pi}{2}\alpha, \quad (10)$$

and converge uniformly to  $f(z)$  in every closed part of  $\Delta_\alpha$ .

3°. We assign to the class  $N_2(\alpha, \omega, \rho, \sigma_0)$  ( $0 < \alpha < 2$ ,  $-1 < \omega < 1$ ,  $\rho > 0$ ) the functions  $f(z)$ , holomorphic in the angle  $\Delta_\alpha$  and satisfying the conditions:

- a)  $f(z)$  has order of growth not exceeding  $\rho \geq \max\{\alpha^{-1}, (2-\alpha)^{-1}\}$ , and type not exceeding  $\sigma_0$ , and on the boundary of  $\Delta_\alpha$  has finite limiting values almost everywhere

$$f(re^{\pm i \frac{\pi}{2}\alpha});$$

- b) the integrals exist and are uniformly bounded

$$\int_0^\infty |f(re^{i\varphi})|^2 r^\omega dr \leq M < +\infty \quad \left(\frac{\pi}{2\rho} \leq |\varphi| \leq \frac{\pi}{2}\alpha\right). \quad (11)$$

**Theorem 4.** 1) If  $f(z) \in N_2(\alpha, \omega, \rho, \sigma_0)$  and is continuous in a neighborhood of  $z = 0$ , then the representation is valid

$$\begin{aligned} f(r^{1/\rho} e^{i\varphi}) r^{\mu-1} &= \frac{d}{dr} \left\{ r^\mu \int_0^\infty E_\rho \left\{ r^{1/\rho} e^{i\varphi} \tau^{-1/\rho} e^{i(\alpha + \frac{1}{\rho}) \frac{\pi}{2}}; \mu + 1 \right\} v_{(-)}(\tau) \tau^{\mu-1} d\tau \right\} \\ &+ \frac{d}{dr} \left\{ r^\mu \int_0^\infty E_\rho \left\{ r^{1/\rho} e^{i\varphi} \tau^{-1/\rho} e^{-i(\alpha + \frac{1}{\rho}) \frac{\pi}{2}}; \mu + 1 \right\} v_{(+)}(\tau) \tau^{\mu-1} d\tau \right\} \\ &+ r^{\mu-1} \int_0^\infty E_\rho \left\{ r^{1/\rho} e^{i\varphi} \tau^{-1/\rho}; \mu \right\} \psi(\tau) \tau^{\mu-1} d\tau \end{aligned} \quad (12)$$

for all  $|\varphi| \leq \frac{\pi}{2}\alpha$ , where

$$\mu = \frac{1 + \rho + \omega}{2\rho}, \quad v_{(\pm)}(\tau) \in L_2(0, +\infty), \quad \psi(\tau) \in L_2(0, \sigma_0);$$

moreover, when  $|\varphi| = \frac{\pi}{2}\alpha$ , equality (12) holds for almost all  $r$ .

( $0 \leq r < +\infty$ ), and for  $z \in \Delta_\alpha$  it can be written more simply as:

$$f(z) = \int_0^\infty E_\rho \left\{ z\tau^{\frac{1}{\rho}} e^{i(\alpha+\frac{1}{\rho})\frac{\pi}{2}}; \mu \right\} v_{(-)}(\tau)\tau^{\mu-1} d\tau +$$

$$+ \int_0^\infty E_\rho \left\{ z\tau^{\frac{1}{\rho}} e^{-i(\alpha+\frac{1}{\rho})\frac{\pi}{2}}; \mu \right\} v_{(+)}(\tau)\tau^{\mu-1} d\tau + \int_0^{\sigma_0} E_\rho \left\{ z\tau^{\frac{1}{\rho}}; \mu \right\} \psi(\tau)\tau^{\mu-1} d\tau,$$
(13)

where the first two integrals on the right converge absolutely.

- 2) If the function  $f(z)$  is defined by equality (12), where  $v_{(\pm)}(\tau) \in L_2(0, \infty)$ ,  $\psi(\tau) \in L_2(0, \sigma_0)$ , then it is holomorphic in  $\Delta_\alpha$ , has order not exceeding  $\rho \geq \max\{\alpha^{-1}; (2-\alpha)^{-1}\}$ , and type not exceeding  $\sigma_0$ , and satisfies conditions (11).

From this there also follows the following approximation theorem.

**Theorem 5.** Let  $f(z) \in N_2(\alpha, \omega, \rho, \sigma_0)$ , and let the entire functions  $f_\sigma(z)$  of order  $\leq \rho$  and type not exceeding  $\sigma = \max\{\sigma_0, \sigma_1\}$  be defined by the formula

$$f_\sigma(z) = \int_0^{\sigma_1} E_\rho \left\{ z\tau^{\frac{1}{\rho}} e^{i(\alpha+\frac{1}{\rho})\frac{\pi}{2}}; \mu \right\} v_{(-)}(\tau)\tau^{\mu-1} d\tau +$$

$$+ \int_0^{\sigma_1} E_\rho \left\{ z\tau^{\frac{1}{\rho}} e^{-i(\alpha+\frac{1}{\rho})\frac{\pi}{2}}; \mu \right\} v_{(+)}(\tau)\tau^{\mu-1} d\tau + \int_0^{\sigma_0} E_\rho \left\{ z\tau^{\frac{1}{\rho}}; \mu \right\} \psi(\tau)\tau^{\mu-1} d\tau,$$
(14)

where the functions  $v_{(\pm)}(\tau)$  are defined by (9), and

$$\psi(\tau) = \frac{e^{i\frac{\pi}{2}(1-\mu)}}{2\pi\rho} \frac{d}{d\tau} \int_0^\infty \frac{e^{it\tau} - 1}{it} f\left(t^{\frac{1}{\rho}} e^{-i\frac{\pi}{2\rho}}\right) t^{\mu-1} dt +$$

$$+ \frac{e^{-i\frac{\pi}{2}(1-\mu)}}{2\pi\rho} \frac{d}{d\tau} \int_0^\infty \frac{e^{-it\tau} - 1}{-it} f\left(t^{\frac{1}{\rho}} e^{i\frac{\pi}{2\rho}}\right) t^{\mu-1} dt.$$
(15)

Then the entire functions  $f_\sigma(z)$  converge in the mean to  $f(z)$  in the sense that

$$\lim_{\sigma \rightarrow \infty} \int_0^\infty |f(re^{i\varphi}) - f_\sigma(re^{i\varphi})|^2 r^\omega dr = 0$$

for all  $|\varphi| \leq \frac{\pi}{2}\alpha$ , and converge uniformly to  $f(z)$  in any closed part of  $\Delta_\alpha$ .

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*

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