THEOREMS ON THE COMPLETENESS OF THE SYSTEM OF EIGEN- AND ASSOCIATED ELEMENTS OF OPERATORS WITH DISCRETE SPECTRUM

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Abstract

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MATHEMATICS

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THEOREMS ON THE COMPLETENESS OF THE SYSTEM OF EIGEN- AND ASSOCIATED ELEMENTS OF OPERATORS WITH DISCRETE SPECTRUM

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Let T be a completely continuous operator acting in a separable Hilbert space $\mathfrak{H}$. The operator T is called an **operator of Hilbert-Schmidt type** (of H.-S. type) if

$$\sum_s (T\varphi_s, T\varphi_s) < \infty, \tag{1}$$

where φ_s $(s = 1, 2, ...)$ is an orthonormal basis in $\mathfrak{H}$. In the case of the integral operator

$$Kf = \int_0^1 k(x,t) f(t)\, dt,$$

which acts in $\mathcal{L}_2(0,1)$ (in the Hilbert space of functions whose square is Lebesgue integrable on the interval $(0,1)$), condition (1) is equivalent to the condition

$$\int_0^1 \int_0^1 |k(x,t)|^2\, dx\, dt < \infty. \tag{2}$$

Theorem 1. *Let an operator T of H.-S. type, acting in a separable Hilbert space $\mathfrak{H}$, be written in the form*

$$T = A + iB, \tag{3}$$

and let the self-adjoint operators A and B be sign-definite (the quadratic forms (Af, f) and (Bf, f) preserve their sign). Then the eigen- and associated elements of the operator T, corresponding to the nonzero points of the spectrum, form a system complete in the range of values of the operator T. If to the *indicated system one adds a basis in the subspace of solutions of the equation $Tf = 0$, then one obtains a system complete in $\mathfrak{H}$.*

For the proof, denote by Q_1 the closed subspace spanned by all eigen- and associated elements of the operator T corresponding to nonzero points of the spectrum. Let Q_2 be the orthogonal complement to Q_1. It is easy to show that the operator T^* adjoint to T is invariant on Q_2. Denote by $T^*_{q_2}$ the operator induced by the operator T^* on Q_2. One can show that zero is the only point of the spectrum of the operator $T^*_{q_2}$ (cf. (^{3})).

After these remarks, let us map Q_2, preserving the scalar product, onto $\mathcal{L}_2(0,1)$ (or onto some subspace of it, if Q_2 is finite-dimensional) and consider in $\mathcal{L}_2(0,1)$ an integral operator K, unitarily equivalent to the operator $T^*_{q_2}$. Such an operator is easy to construct. Let

$D^*(\lambda)$ is the Fredholm determinant of the integral operator K ((1), p. 196):

$$D^*(\lambda) = 1 + \sum_{n=2}^{\infty} \frac{(-\lambda)^n}{n!} \delta_n, \tag{4}$$

where

$$\delta_n = \int_0^1 \cdots \int_0^1 \begin{vmatrix} 0 & k(x_1,x_2) & \cdots & k(x_1,x_n) \\ k(x_2,x_1) & 0 & \cdots & k(x_2,x_n) \\ \cdots & \cdots & \cdots & \cdots \\ k(x_n,x_1) & \cdots & \cdots & 0 \end{vmatrix} dx_1 \cdots dx_n. \tag{5}$$

As Carleman showed ((1), p. 217), $D^*(\lambda)$ is an entire function for which the representation

$$D^*(\lambda) = 1 \cdot \prod_{s=1}^{\infty} (1 - \lambda\lambda_s) e^{\lambda\lambda_s} \tag{6}$$

is valid.

Here λ_s denote the eigenvalues of the operator. Since in our case the operator K has no eigenvalues different from zero, on the basis of (6) we conclude that $D^*(\lambda) \equiv 1$.

From formulas (4) and (5), then, in particular, it follows that

$$\delta_2 = -\int_0^1 \int_0^1 k(x_1,x_2)k(x_2,x_1)\, dx_1 dx_2 = 0. \tag{7}$$

On the other hand, $-\operatorname{Im}\delta_2$ is the trace of the imaginary part* of the operator K^2. Thus, $\operatorname{Sp}(\operatorname{Im} K^2) = 0$. Hence, by unitary equivalence, we conclude that also

$$\mathrm{Sp}(\mathrm{Im}\, T_{q_2}^{*2}) = 0. \tag{8}$$

Represent the operator $T_{q_2}^*$ in the form $T_{q_2}^* = A_{q_2} - iB_{q_2}$, where A_{q_2} and B_{q_2} are self-adjoint operators defined in Q_2, and compute the left-hand side of (8) in an orthonormal basis of eigenvectors of the operator B_{q_2}. After simple transformations we obtain

$$\mathrm{Sp}(\mathrm{Im}\, T_{q_2}^{*2}) = -\sum_s 2\mu_s \,(A_{q_2}\psi_s, \psi_s) = 0, \tag{9}$$

where μ_s $(s = 1, 2, ...)$ are the eigenvalues of B_{q_2}.

The operators A_{q_2} and B_{q_2} are sign-definite together with A and B in (3). Therefore from equality (9) we conclude that

$$\mu_s(A_{q_2}\psi_s, \psi_s) = 0 \tag{10}$$

for all $s = 1, 2, ...$ We shall show that all eigenvalues of the operator B_{q_2} are equal to zero.

Suppose the contrary. Let $\mu_s^0 \neq 0$. Then from (10) it follows that $(A_{q_2}\psi_s^0, \psi_s^0) = 0$, where ψ_s^0 is the eigen-element of B_{q_2} corresponding to μ_s^0. In view of the sign-definiteness of A_{q_2}, this means that $A_{q_2}\psi_s^0 = 0$. Applying further the operator $T_{q_2}^*$ to ψ_s^0, we shall have

$$T_{q_2}^*\psi_s^0 = A_{q_2}\psi_s^0 - iB_{q_2}\psi_s^0 = -i\mu_s^0\psi_s^0. \tag{11}$$

* By the imaginary part of an operator C is meant the operator $\frac{1}{2i}(C - C^*)$, denoted by $\mathrm{Im}\, C$.

Equality (11) contains a contradiction, since, by hypothesis, $T_{q_2}^*$ has only the zero point of the spectrum, while $\mu_s^0 \neq 0$. Thus all $\mu_s = 0$ $(s = 1, 2, ...)$. Consequently, $B_{q_2} = 0$. Similarly one can show that $A_{q_2} = 0$. As a result, the operator $T_{q_2}^* = 0$, and this already means completeness in the range of values of the operator T. Indeed, if $g \in Q_2$, then $(Th, g) = (h, T^*g) = (h, T_{q_2}^*g) = 0$; consequently, for any $h \in \mathfrak{H}$ the element Th is orthogonal to Q_2, and hence $Th \in Q_1$.

The validity of the remark at the end of the theorem follows from the fact that the operator T, just as T^*, maps Q_2—the orthogonal complement to Q_1—into zero.* The theorem is proved.

It should be noted that examples can be given which demonstrate the essential nature of the definiteness conditions put forward in Theorem 1.

From Theorem 1 the following theorem can be obtained:

Theorem 2. Let the operators L_1 and L_2 be symmetric on some dense manifold $\mathfrak{D}$ in $\mathfrak{H}$, and suppose that for some λ the manifold $(L_1 + iL_2 - \lambda E)\mathfrak{D}$ is dense in $\mathfrak{H}$.

Suppose, further, that one of the following two conditions is satisfied:

a) The operators L_1 and L_2 are semibounded on $\mathfrak{D}$ (both, for definiteness, from below) and, in addition, one of the three self-adjoint operators $\widetilde{L}_1$, $\widetilde{L}_2$, or $\widetilde{L_1 + L_2}$ has a resolvent of type Γ.–III.**

b) For all $f \in \mathfrak{D}$

$$(L_1 f, f) - |(L_2 f, f)| \geq -\gamma^2 (f, f)$$

and, in addition, the operator $\widetilde{L}_1$ has a resolvent of type Γ.–III.

Then the non-self-adjoint operator $L = L_1 + iL_2$ admits a closure $\widetilde{L}$; the operator $\widetilde{L}$ has a resolvent of type Γ.–III., and the system of its eigen and associated elements is complete in $\mathfrak{H}$.

Theorems 1 and 2 can be used in the study of various integral and differential operators. We give two examples.

Example 1 (cf. (2^{-4})). Consider a strongly elliptic operator (5):

$$Lu = \sum_{[i,j]} \frac{\partial^m}{\partial x_{i_1} \cdots \partial x_{i_m}} C^{[i,j]}(x) \frac{\partial^m u}{\partial x_{j_1} \cdots \partial x_{j_m}} +$$

$$+ \sum_{[i,j]} \frac{\partial^m}{\partial x_{i_1} \cdots \partial x_{i_m}} K^{[i,j]}(x) \frac{\partial^m u}{\partial x_{j_1} \cdots \partial x_{j_m}} + Pu,$$

which acts on the manifold $\mathfrak{D}$ of vector-functions $u(x)$ satisfying, on the boundary Γ of a certain bounded domain G of n-dimensional Euclidean space, the boundary conditions

$$u(x)\big|_\Gamma = \frac{\partial u}{\partial \nu}\bigg|_\Gamma = \cdots = \frac{\partial^{m-1} u}{\partial \nu^{m-1}}\bigg|_\Gamma = 0,$$

where ν is the normal to Γ. For brevity we write the operator L in the form:

$$Lu = Cu + Ku + Pu,$$

where by C and K we have denoted, respectively, the semibounded symmetric and skew-symmetric operators of order $2m$, and by P an arbitrary operator of order $< 2m$.

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* Indeed, if $g \in Q_2$, then $T^*g = Ag - iBg = 0$, and hence, by the definiteness of A and B: $Ag = 0$ and $Bg = 0$, whence $Tg = 0$.

** A wavy line denotes a self-adjoint extension of the corresponding operator.

Let $4m - n > 0$, and suppose that for all $u \in \mathfrak{D}$

$$(Cu, u) - |(Ku, u)| \geq -\gamma^2 (u, u). \tag{12}$$

Then the system of eigen- and associated elements of the operator L is complete in $\mathcal{L}_2(G)$. If the operator iK is semibounded, then condition (12) may be omitted.

Example 2 (cf. ($6^{--}8$)). Consider the operator

$$Ly = -y'' + (q(x) + ir(x))y, \tag{13}$$

defined on some set dense in $\mathcal{L}_2(-\infty, +\infty)$. Let the function $q(x)$ be bounded below, and let $r(x)$ be semibounded (i.e., bounded either above or below). Then, for completeness of the system of eigen- and associated elements of the operator L in $\mathcal{L}_2(-\infty, +\infty)$, it is sufficient that, for some $\alpha > 2/3$,

$$\lim_{|x| \to \infty} \frac{q(x) + |r(x)|}{|x|^\alpha} \geq C > 0.$$

We emphasize that this thereby establishes completeness, in the case of operators of the form (13), also when the "imaginary potential" $r(x)$ tends to infinity.*

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REFERENCES

1. T. Carleman, Math. Zs., **9**, 196 (1921).
2. M. V. Keldysh, DAN, **77**, 11 (1951).
3. F. E. Browder, Proc. Nat. Acad. Sci. USA, **39**, 433 (1953).
4. M. Naimark, DAN, **112**, 198 (1957).

5. M. L. Vishik, Mat. sborn., **29** (71), 3, 615 (1951).
6. M. A. Naimark, DAN, **98**, 727 (1954).
7. J. Schwartz, Pacific J. Math., **4**, 2, 415 (1954).
8. V. B. Lidskii, DAN, **110**, 172 (1956).

* The corresponding problem was posed by I. M. Gel' fand.

Note: Figure translations are in progress. See original paper for figures.

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