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**Abstract**

**Full Text**

**S. I. Adian**

**ON ALGORITHMIC PROBLEMS IN EFFECTIVELY COMPLETE CLASSES OF GROUPS**

*(Presented by Academician I. M. Vinogradov, 16 V 1958)*

The purpose of the present note is a further generalization of the theorems on undecidable algorithmic problems of group theory <sup>(1-3)</sup>.\* Here we use the notation of papers <sup>(2)</sup>, <sup>(3)</sup>. In the first part of the note we give a number of remarks on paper <sup>(3)</sup>, making it possible to reconstruct completely the proof of the main lemma of that paper.\*\* From the remarks given below there follows the proof of a somewhat stronger assertion, from which the indicated lemma follows as a consequence. Namely, in the first part the following lemma on the groups  $F_{qA}$  will be proved.

**Lemma B.** *Let the word  $A$  of the group  $F$  contain no letters  $p^\sigma$ . If  $A = 1$  in the group  $F$ , then  $F_{qA}$  is the trivial group. If  $A \neq 1$  in  $F$ , then the group  $F$  is a subgroup of the group  $F_{qA}$ .*

In the second part, using Lemma B, we shall prove some theorems on undecidable algorithmic problems for effectively complete classes of finitely presented groups (briefly: e.-c. groups).

**I. Some remarks on paper <sup>(3)</sup>.** The analogy between the defining relations of the groups  $\mathfrak{A}_{q,A,B}$ , constructed in <sup>(2)</sup>, and the defining relations of the groups  $F_{qA}$ , constructed in <sup>(3)</sup>, is evident. In the defining relations of the groups  $F_{qA}$ , the role of the words  $AB^{-1} \equiv pXpX^+Y^{+1}p^{-1}Y^{-1}p^{-1}$  is played by the words  $E \equiv p^2Ap^{-1}Ap^{-1}$ . This analogy is deep.

The first two paragraphs of Chapter II of paper <sup>(2)</sup> are devoted to the proof of Lemma 4, § 2, Chapter II. In doing so, a number of auxiliary assertions are used, which are contained in Chapter I. The assertions contained in § 1, Chapter I, are of a general character and may be applied to any group. As for the assertions contained in § 2, Chapter I of paper <sup>(2)</sup>, they concern the particular group  $\mathfrak{A}_p$  and are connected with the features of the construction of this group. Nevertheless, analogous assertions are true for the group  $F = F_0 * F_2' * F_3$  constructed in <sup>(3)</sup>.

For each of the assertions of § 2, Chapter I of paper <sup>(2)</sup>, used in the first two paragraphs of Chapter II, we indicate the corresponding assertion about the group  $F$ , which will be used in the proof of Lemma B.

- 1) Lemma 1, § 2, Chapter I (p. 242). To this lemma there corresponds the following obvious assertion.

**Lemma 1\*.** *Let  $D$  be a word of the group  $F$ , containing no letters  $p^\sigma$ . If  $D = 1$  in  $F$ , then in the group  $F$  the projections of the word  $D$  onto the alphabets of the groups  $F_0$  and  $F'_2$  are also equal to 1.*

- 2) Lemma 3, § 2, Chapter I (p. 243). To this lemma there corresponds Lemma 3 of paper <sup>(3)</sup> (see p. 10). This lemma is also obvious.

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\* In <sup>(3)</sup>, on p. 10, after formula (8) there should follow the line: “where by  $E$  is denoted the word  $p^2Ap^{-1}Ap^{-1}$ .”

\*\* Theorem 1 of paper <sup>(3)</sup> was also proved by Rabin in <sup>(5)</sup>.

- 3) Lemma 4, § 2, Ch. I (p. 244). This lemma will be replaced by Lemma 2 of paper <sup>(3)</sup> (p. 10). The proof of Lemma 2 of paper <sup>(3)</sup> rests on Lemma 1 of the same paper, which we shall first prove.

**Proof of Lemma 1.** Let a c.-p. group  $F_2$  be given by generators  $a_1, a_2, \dots, a_n$  and defining relations  $A_i = 1$ . As the group  $F'_2$  one may take the group defined by the generators

$$b_1, b_2, \dots, b_n, c \tag{1}$$

and defining relations  $B_i = 1$ , where the word  $B_i$  is obtained from the word  $A_i$  by replacing each letter  $a_j^\sigma$ ,  $\sigma = \pm 1$ , occurring in  $A_i$ , by the word  $(cb_j)^\sigma$ . The subgroup of  $F'_2$  generated by all the elements  $cb_j$  is isomorphic to the group  $F_2$ . It is also easy to see that the generators (1) have infinite order in  $F'_2$ .

**Proof of Lemma 2.** All generators of the group  $F$  have infinite order in  $F$  (for the generators of the group  $F_0$  this follows from Theorem 2, p. 69, of paper <sup>(4)</sup>, and for the generators of the group  $F'_2$  from the lemma just proved; the letter  $p$  does not occur in the defining relations of the group  $F$ ). Any two distinct letters of the alphabet of the group  $F$  are generators of different free factors of the group  $F$ . Consequently, they will be free generators of the subgroup generated by them in  $F$ . Lemma 2 is proved.

- 4) Lemma 2', § 2, Ch. I (p. 245). This lemma is a consequence of the fact that  $a_n$  and  $a_{n-1}$  are free generators of the subgroup generated by them in  $\mathfrak{A}_p$ . By Lemmas 3 and 2 of paper <sup>(3)</sup>,  $e_k$  and  $e_{k-1}$  are free generators of the subgroup generated by them in  $F$ . Hence it follows that in  $F$ , for  $e_k$  and  $e_{k-1}$ , the assertion of Lemma 2' is true.

- 5) Lemma 7, § 2, Ch. I (p. 254) may be replaced by the following assertion.

**Lemma 7\*.** *Let  $A, C, D$  be arbitrary words of the group  $F$  not containing letters  $p^\sigma$ , and let  $A \neq 1$  in  $F$ . If, moreover,*

$$p^2Ap^{-1}Ap^{-1}DpA^{-1}pA^{-1}p^{-2}C = 1 \quad \text{in } F,$$

then in the group  $F$  the word  $C$  and also the projections of the word  $D$  onto the alphabets of the groups  $F_0$  and  $F'_2$  are equal to 1.

This lemma is a consequence of the fact that  $F_0$ ,  $F'_2$ , and  $F_3 = \{p\}$  are free factors of the group  $F$ .

6) Lemma 6, § 2, Ch. I, and its corollary (pp. 246 and 253) are replaced by the following lemma.

**Lemma 6\***. *Let the word  $A$  of the group  $F$  contain no letters  $p$  and not be equal to 1 in  $F$ . In order that the word*

$$Q \equiv e^{k_1}(p^2 Ap^{-1} Ap^{-1})^{s_1} e^{k_2} \dots e^{k_\mu}(p_2 Ap^{-1} Ap^{-1})^{s_\mu} e^{k_{\mu+1}},$$

where  $e$  is an arbitrary letter of the alphabet of the group  $F$ , be equal to 1 in  $F$ , it is necessary and sufficient that the corresponding sequence of numbers  $k_1, s_1, k_2, \dots, k_\mu, s_\mu, k_{\mu+1}$  be degenerate.

Lemma 6\* is proved in the same way as Lemma 6 and its corollary were proved in Ch. I of paper (2).

Thus, for each of the auxiliary assertions concerning the group  $\mathfrak{A}_p$  (1)–(6)), we have the corresponding assertion concerning the group  $F$ . Now the proof of Lemma B is obtained by a literal repetition of all the arguments of the first two paragraphs of Chapter II of paper (2).

## II. Algorithmic problems in effectively complete classes of groups.

By the recognition problem for a group property  $\alpha$  for groups of a class  $K$  we shall mean the problem of finding an algorithm which, for any group of the class  $K$ , would decide whether it has the property  $\alpha$ .

or not. In works (1–3) a special case of such problems was studied, when the class  $K$  coincides with the class of all f.p. groups.

We shall call a class of f.p. groups  $K$  **effectively complete** if there exists an algorithm  $\mathfrak{A}$  which transforms a presentation of an arbitrary f.p. group  $F$  into a presentation of such a group from the class  $K$ , to some subgroup of which the group  $F$  is isomorphic. In this case we shall call the algorithm  $\mathfrak{A}$  an **algorithm effectively realizing the completeness of the class  $K$** . We shall say that the algorithm  $\mathfrak{A}$  **leaves the group  $F_0$  fixed** if it transforms any presentation of the group  $F_0$  into a presentation of this same group.

**Condition T.** We shall say that for the pair  $(K, \alpha)$ , where  $K$  is some class of f.p. groups and  $\alpha$  is any invariant group-theoretic property, condition T is satisfied if one can indicate an f.p. group  $F_1$  from the class  $K$  possessing the property  $\alpha$ , and such an f.p. group  $F_2$  which cannot be embedded in any f.p. group with the property  $\alpha$ .

**Theorem 1.** *Let  $K$  be some effectively complete class of f.p. groups. If the invariant property  $\alpha$  is such that condition T is satisfied for the pair  $(K, \alpha)$ , and*

the algorithm  $\mathfrak{A}$ , effectively realizing the completeness of the class  $K$ , leaves fixed the group  $F_1$  with the property  $\alpha$ , then the problem of recognizing the property  $\alpha$  for groups of the class  $K$  is undecidable.

**Proof.** Consider the group  $F_2$ , which by condition T is not embeddable in any f.p. group with the property  $\alpha$ . Using Lemma 1 of the paper <sup>(3)</sup>, embed  $F_2$  in an f.p. group  $F'_2$ , all generators of which have infinite order. Next, on the basis of the group  $F = F_0 * F'_2 * \{p\}$ , where  $F_0$  is an f.p. group with unsolvable identity problem, we construct, exactly as indicated in <sup>(3)\*</sup>, a system of groups  $F_{qA}$ . Above we proved that Lemma B is valid for the groups  $F_{qA}$ . Let  $F_A = F_{qA} * F_1$ , where  $F_1$  is a group of the class  $K$  possessing the property  $\alpha$ . Consider the system of groups  $G_A = \mathfrak{A}(F_A)$ , where  $A$  is an arbitrary word of the group  $F_0$ . Since the algorithm  $\mathfrak{A}$  effectively realizes the completeness of the class  $K$ , all the groups  $G_A$  belong to the class  $K$ . Theorem 1 will be proved if we prove that there is no algorithm which would decide, for every group  $G_A$ , whether it has the property  $\alpha$  or not. For this it is enough to prove that  $G_A$  will have the property  $\alpha$  if and only if  $A = 1$  in the group  $F_0$ .

Let  $A = 1$  in  $F_0$ . Then  $A = 1$  in  $F$ . By Lemma B, in this case the group  $F_{qA}$  is trivial. Consequently,  $F_A$  is isomorphic to the group  $F_1$ , and hence  $G_A$  is isomorphic to  $F_1$ . Then the invariant property  $\alpha$  will hold in  $G_A$ .

Let  $A \neq 1$  in  $F_0$ . Then  $A \neq 1$  in  $F$ . In this case, by Lemma B, the group  $F$  is a subgroup of the group  $F_{qA}$ . We have

$$F_2 \subset F'_2 \subset F \subset F_{qA} \subset F_A \subset \mathfrak{A}(F_A) = G_A,$$

i.e. the group  $F_2$  is embedded in  $G_A$ . Then, by condition T, the group  $G_A$  cannot have the property  $\alpha$ . The theorem is proved.

The following special case of Theorem 1 is of interest.

**Theorem 2.** *Let  $K$  be an effectively complete class of f.p. groups whose effective algorithm leaves fixed every group of this class. If the invariant property  $\alpha$  is such that condition T is satisfied for the pair  $(K, \alpha)$ , then the problem of recognizing the property  $\alpha$  for groups of the class  $K$  is undecidable.*

It is easy to see that Theorem 1 of the paper <sup>(3)</sup> is a special case of this theorem.

We shall call a class of f.p. groups  $K$  **recursive** if there exists an algorithm which makes it possible, for any f.p. group, to decide whether it belongs to the class  $K$  or not.

**Theorem 3.** *Let  $K$  be a recursive and effectively complete class of f.p. groups, and let  $\alpha$  be some invariant group-theoretic property. If for the pair  $(K, \alpha)$  condition T is satisfied, then the problem of recognizing the property  $\alpha$  for groups of the class  $K$  is undecidable.*

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\* See the footnote on p. 13.

**Proof.** Let  $\mathfrak{N}$  be an effectivizing algorithm for the class  $K$ , and let  $\mathfrak{M}$  be an algorithm which makes it possible to check, for any finitely generated group, whether it belongs to the class  $K$  or not. Using these two algorithms, we easily construct an algorithm  $\mathfrak{N}^*$  which effectivizes the completeness of the class  $K$  and leaves fixed every group of the class  $K$ , thereby proving Theorem 3.

As a nontrivial example of a recursive and effectively complete class of finitely generated groups one may indicate the class of finitely generated groups that coincide with their commutator subgroup.

Every system of nontrivial identical relations

$$L_i \equiv 1 \quad (i = 1, 2, \dots, k), \quad (2)$$

written with the aid of variable symbols  $x_1, x_2, \dots, x_r$ , determines a class of groups that are conditionally trivial with respect to this system of identical relations (see (3), p. 11).

**Theorem 4.** *Let  $K_1$  be the class of groups conditionally trivial with respect to the system of nontrivial identical relations (2). If an invariant group-theoretic property  $\alpha$  is such that the pair  $(K_1, \alpha)$  satisfies condition T, then the problem of recognizing the property  $\alpha$  for groups of the class  $K_1$  is unsolvable.*

**Proof.** First embed the group  $F_2$ , by Lemma 1 of [3], in a group  $F'_2$ , all generators of which have infinite order. At the same time we shall take the group  $F'_2$  so that it contains a free subgroup with two generators  $b_1$  and  $b_2$ . Let  $B_1, B_2, \dots, B_r$  be free generators of the subgroup generated by them in the free group with generators  $b_1$  and  $b_2$ . Substitute in the words  $L_i$  everywhere in place of the variables  $x_1, x_2, \dots, x_r$  the words  $B_1, B_2, \dots, B_r$ . Denote the resulting words of the group  $\{b_1, b_2\}$  by  $Y_i$  ( $i = 1, 2, \dots, k$ ). Since all the identical relations (2) are nontrivial by hypothesis, no  $Y_i$  is equal to 1 in the group  $\{b_1, b_2\}$ . Hence  $Y_i \neq 1$  also in  $F'_2$ . Let  $C$  be an arbitrary word of the group  $F_0$ . Put

$$A = [\dots [[C, Y_1], Y_2] \dots Y_k], \quad (3)$$

where the square brackets denote the taking of a commutator.  $A$  is a word of the group

$$F = F_0 * F'_2 * \langle \rho \rangle.$$

It is easy to see that  $A = 1$  in  $F$  if and only if  $C = 1$  in  $F_0$ . Therefore there is no algorithm which would check, for any word  $A$  of the form (3), whether it is equal to 1 in  $F$  or not. Note also that adding to the relations of the group  $F$  any one of the identical relations (2) turns the corresponding word  $Y_i$  into 1, and together with it also the word  $A$ .

Now consider the system of groups  $F_{q_A}$ , constructed in the same way as was done in [3], which correspond to the words (3) for all possible  $C \in F_0$ . By Lemma B, each of these groups will be conditionally trivial with respect to the system (2), i.e. all these groups belong to the class  $K_1$ . Obviously, every group

$$F_A = F_{q_A} * F_1,$$

where  $F_1$  is a group which, by condition T, belongs to  $K_1$  and has property  $\alpha$ , also belongs to the class  $K_1$ . Theorem 4 will be proved if we prove that there is no algorithm which would check, for any group  $F_A$ , whether it has property  $\alpha$  or not. And for this it is enough to prove that the group  $F_A$  has property  $\alpha$  if and only if  $A = 1$  in the group  $F$ . This is proved in the same way as in Theorem 1.

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*Note: Figure translations are in progress. See original paper for figures.*

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