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Abstract

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THEORY OF ELASTICITY

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STABILITY OF ELASTIC-PLASTIC INHOMOGENEOUS SHELLS

(Presented by Academician L. I. Sedov, 18 XI 1957)

Below, the purely plastic loss of stability of inhomogeneous shells is considered. For definiteness, in calculating the stiffnesses a bimetallic shell is taken. Physical relations for bimetallic shells are obtained, and stability equations are given for shallow shells of arbitrary form with respect to the deflection and to the force function satisfying the equilibrium equations of the plane problem. These results are valid for the theory of plastic flow and the deformation theory; in both cases the compressibility of the material is taken into account. A formula is given for determining the critical forces of local loss of stability of shells under combined loads. From it several particular results are obtained.

1. Variations of the specific forces in the shell. All quantities pertaining to the outer layer of the shell will be supplied with a prime, and those pertaining to the inner layer—with a double-prime. Then the variations of the specific normal forces, shear force, bending and twisting moments have the form

$$\begin{aligned} \delta N_1 &= \int_0^{h'} \delta\sigma_x'\,dz + \int_{-h''}^{0} \delta\sigma_x''\,dz, & \delta N_2 &= \int_0^{h'} \delta\sigma_y'\,dz + \int_{-h''}^{0} \delta\sigma_y''\,dz, \\ \delta T &= \int_0^{h'} \delta\tau_{xy}'\,dz + \int_{-h''}^{0} \delta\tau_{xy}''\,dz, & \delta M_1 &= \int_0^{h'} \delta\sigma_x' z\,dz + \int_{-h''}^{0} \delta\sigma_x'' z\,dz, \\ \delta M_2 &= \int_0^{h'} \delta\sigma_y' z\,dz + \int_{-h''}^{0} \delta\sigma_y'' z\,dz, & \delta H &= \int_0^{h'} \delta\tau_{xy}' z\,dz + \int_{-h''}^{0} \delta\tau_{xy}'' z\,dz. \end{aligned} \tag{1}$$

where $\delta\sigma_x,\ \delta\sigma_y,\ \delta\tau_{xy}$ are the stresses in the shell due to loss of stability; h is the thickness; z is the distance through the thickness from the surface of bonding of the shell.

If one is based on the Kirchhoff-Love hypotheses, then the variations of stresses for a purely plastic state are equal to ([1]) (index omitted)

$$
\begin{aligned}
\delta\sigma_x &= \frac{1}{h}\left[b_{11}\delta\varepsilon_1 + b_{12}\delta\varepsilon_2 - b_{13}\delta\gamma_{12} - z\left(b_{11}\delta\chi_1 + b_{12}\delta\chi_2 - 2b_{13}\delta\chi_{12}\right)\right], \\
\delta\sigma_y &= \frac{1}{h}\left[b_{21}\delta\varepsilon_1 + b_{22}\delta\varepsilon_2 - b_{23}\delta\gamma_{12} - z\left(b_{21}\delta\chi_1 + b_{22}\delta\chi_2 - 2b_{23}\delta\chi_{12}\right)\right], \qquad (2) \\
\delta\tau_{xy} &= \frac{1}{h}\left[-b_{31}\delta\varepsilon_1 - b_{32}\delta\varepsilon_2 + b_{33}\delta\gamma_{12} - z\left(b_{31}\delta\chi_1 + b_{32}\delta\chi_2 - 2b_{33}\delta\chi_{12}\right)\right].
\end{aligned}
$$

Here $\delta\varepsilon_1,\ \delta\varepsilon_2,\ \delta\gamma_{12}$ are, respectively, the linear and angular strains of the surface of bonding during buckling; $\delta\chi_1,\ \delta\chi_2,\ \delta\chi_{12}$ are parameters of change

of the curvatures of the interface surface under buckling; $b_{11},\ b_{12} = b_{21},\ b_{22},\ b_{23} = b_{32}, \dots, \dots, b_{33}$ are coefficients depending on the theory of plasticity and determined by the mechanical characteristics of the material (for the deformation theory and the theory of plastic flow they have in the general case been computed in (1)).

Substituting (2) into (1), we obtain:

$$
\begin{aligned}
\delta N_1 &= B_{11}\delta\varepsilon_1 + B_{12}\delta\varepsilon_2 - B_{13}\delta\gamma_{12} - B_{14}\delta\chi_1 - B_{15}\delta\chi_2 + B_{16}\delta\chi_{12}, \\
\delta N_2 &= B_{21}\delta\varepsilon_1 + B_{22}\delta\varepsilon_2 - B_{23}\delta\gamma_{12} - B_{24}\delta\chi_1 - B_{25}\delta\chi_2 + B_{26}\delta\chi_{12}, \\
\delta T &= -B_{31}\delta\varepsilon_1 - B_{32}\delta\varepsilon_2 + B_{33}\delta\gamma_{12} + B_{34}\delta\chi_1 + B_{35}\delta\chi_2 - B_{36}\delta\chi_{12}, \qquad (3) \\
\delta M_1 &= B_{41}\delta\varepsilon_1 + B_{42}\delta\varepsilon_2 - B_{43}\delta\gamma_{12} - B_{44}\delta\chi_1 - B_{45}\delta\chi_2 + B_{46}\delta\chi_{12}, \\
\delta M_2 &= B_{51}\delta\varepsilon_1 + B_{52}\delta\varepsilon_2 - B_{53}\delta\gamma_{12} - B_{54}\delta\chi_1 - B_{55}\delta\chi_2 + B_{56}\delta\chi_{12}, \\
\delta H &= -B_{61}\delta\varepsilon_1 - B_{62}\delta\varepsilon_2 + B_{63}\delta\gamma_{12} + B_{64}\delta\chi_1 + B_{65}\delta\chi_2 - B_{66}\delta\chi_{12};
\end{aligned}
$$

where

$$
\begin{gathered}
B_{11} = b'_{11} + b''_{11}, \quad B_{12} = B_{21} = b'_{12} + b''_{12}, \quad B_{13} = B_{31} = b'_{13} + b''_{13}, \\
B_{14} = B_{41} = \tfrac{1}{2}(h'b'_{11} - h''b''_{11}), \quad B_{15} = B_{51} = \tfrac{1}{2}(h'b'_{21} - h''b''_{21}), \quad B_{16} = 2B_{61} = h'b'_{13} - h''b''_{13}, \\
B_{22} = b'_{22} + b''_{22}, \quad B_{23} = B_{32} = b'_{23} + b''_{32}, \quad B_{24} = B_{42} = \tfrac{1}{2}(h'b'_{21} - h''b''_{21}), \\
B_{25} = B_{52} = \tfrac{1}{2}(h'b'_{22} - h''b''_{22}), \quad B_{26} = 2B_{62} = h'b'_{32} - h''b''_{32}, \quad B_{33} = b'_{33} + b''_{33}, \\
B_{34} = B_{43} = \tfrac{1}{2}B_{16} = \tfrac{1}{2}(h'b'_{13} - h''b''_{13}), \quad B_{35} = B_{53} = \tfrac{1}{2}(h'b'_{23} - h''b''_{23}), \\
B_{36} = 2B_{63} = h'b'_{33} - h''b''_{33}, \quad B_{44} = \tfrac{1}{3}({h'}^2 b'_{11} + h''^2 b''_{11}), \\
B_{45} = B_{54} = \tfrac{1}{3}({h'}^2 b'_{12} + h''^2 b''_{12}), \quad B_{46} = 2B_{64} = \tfrac{2}{3}({h'}^2 b'_{13} + h''^2 b''_{13}), \\
B_{55} = \tfrac{1}{3}({h'}^2 b'_{22} + h''^2 b''_{22}), \quad B_{56} = 2B_{65} = \tfrac{2}{3}({h'}^2 b'_{23} + h''^2 b''_{23}), \\
B_{66} = \tfrac{2}{3}({h'}^2 b'_{33} + h''^2 b''_{33}).
\end{gathered}
$$

From the first three equations (3) let us express $\delta\varepsilon_1,\ \delta\varepsilon_2,\ \delta\gamma_{12}$ in terms of $\delta N_1,\ \delta N_2,\ \delta T$. We obtain:

$$
\begin{aligned}
\delta\varepsilon_1 &= A_{11}\delta N_1 + A_{12}\delta N_2 - A_{13}\delta T + A_{14}\delta\chi_1 + A_{15}\delta\chi_2 - A_{16}\delta\chi_{12},\\
\delta\varepsilon_2 &= A_{21}\delta N_1 + A_{22}\delta N_2 - A_{23}\delta T + A_{24}\delta\chi_1 + A_{25}\delta\chi_2 - A_{26}\delta\chi_{12},\\
\delta\gamma_{12} &= -A_{31}\delta N_1 - A_{32}\delta N_2 + A_{33}\delta T - A_{34}\delta\chi_1 - A_{35}\delta\chi_2 + A_{36}\delta\chi_{12},
\end{aligned}
\tag{4}
$$

where

$$
A_{11} = \frac{B_{22}B_{33} - B_{23}^2}{\Delta}, \quad A_{12} = A_{21} = \frac{B_{13}B_{23} - B_{12}B_{33}}{\Delta}, \quad A_{13} = A_{31} = \frac{B_{12}B_{23} - B_{13}B_{22}}{\Delta},
$$

$$
A_{22} = \frac{B_{11}B_{33} - B_{13}^2}{\Delta}, \quad A_{23} = A_{32} = \frac{B_{12}B_{13} - B_{11}B_{23}}{\Delta}, \quad A_{33} = \frac{B_{11}B_{22} - B_{12}^2}{\Delta},
$$

$$
\begin{aligned}
A_{14} &= B_{14}A_{11} + B_{24}A_{12} + B_{34}A_{13}, \quad A_{15} = B_{15}A_{11} + B_{25}A_{12} + B_{35}A_{13}, \quad A_{16} = B_{16}A_{11} + B_{26}A_{12} + B_{36}A_{13},\\
A_{24} &= B_{14}A_{12} + B_{24}A_{22} + B_{34}A_{23}, \quad A_{25} = B_{15}A_{12} + B_{25}A_{22} + B_{35}A_{23}, \quad A_{26} = B_{16}A_{12} + B_{26}A_{22} + B_{36}A_{23},\\
A_{34} &= B_{14}A_{13} + B_{24}A_{23} + B_{34}A_{33}, \quad A_{35} = B_{15}A_{13} + B_{25}A_{23} + B_{35}A_{33}, \quad A_{36} = B_{16}A_{13} + B_{26}A_{23} + B_{36}A_{33},
\end{aligned}
$$

and

$$
\Delta = B_{33}(B_{11}B_{22} - B_{12}^2) - B_{11}B_{23}^2 + 2B_{12}B_{13}B_{23} - B_{13}^2B_{22}.
$$

Substituting expressions (4) into the last three formulas (3):

$$
\begin{aligned}
\delta M_1 &= C_{11}\delta N_1 + C_{12}\delta N_2 - C_{13}\delta T - D_{11}\delta\chi_1 - D_{12}\delta\chi_2 + D_{13}\delta\chi_{12},\\
\delta M_2 &= C_{21}\delta N_1 + C_{22}\delta N_2 - B_{23}\delta T - D_{21}\delta\chi_1 - D_{22}\delta\chi_1 + D_{23}\delta\chi_{12},\\
\delta H &= -C_{31}\delta N_1 - C_{32}\delta N_2 + C_{33}\delta T + D_{31}\delta\chi_1 + D_{32}\delta\chi_2 - D_{33}\delta\chi_{12},
\end{aligned}
\tag{5}
$$

where

$$
\begin{gathered}
C_{11} = A_{14}, \quad C_{12} = A_{24}, \quad C_{13} = A_{34}, \quad C_{21} = A_{15}, \quad C_{22} = A_{25}, \quad C_{23} = A_{35},\\
C_{31} = \tfrac{1}{2}A_{16}, \quad C_{32} = \tfrac{1}{2}A_{26}, \quad C_{33} = \tfrac{1}{2}A_{36},\\
D_{11} = B_{44} + B_{41}A_{14} + B_{42}A_{24} + B_{43}A_{34},\\
D_{12} = B_{45} + A_{41}A_{15} + B_{42}A_{25} + B_{43}A_{35},\\
D_{13} = B_{46} - B_{41}A_{16} - B_{42}A_{26} - B_{43}A_{36},\\
D_{21} = B_{54} - B_{51}A_{14} - B_{52}A_{24} - B_{53}A_{34},\\
D_{22} = B_{55} - B_{51}A_{15} - B_{52}A_{25} - B_{53}A_{35},\\
D_{23} = B_{56} - B_{51}A_{16} - B_{52}A_{26} - B_{53}A_{36},\\
D_{31} = B_{64} - B_{61}A_{14} - B_{62}A_{24} - B_{63}A_{34},\\
D_{32} = B_{55} - B_{61}A_{15} - B_{62}A_{25} - A_{63}A_{35}, \quad D_{32} = B_{66} - B_{61}A_{16} - B_{62}A_{26} - B_{63}A_{36}.
\end{gathered}
$$

2. Stability equations

Since the relation between forces and deformations has been established, it is necessary to add to them the equations of neutral equilibrium of a shell element in terms of forces and the compatibility equation. We shall consider the case of a shallow shell. Here the equations of neutral equilibrium are

$$(\delta N_1)_y + (\delta T)_x = 0, \qquad (\delta N_2)_x + (\delta T)_y = 0,$$

$$(\delta M_1)_{xx} + 2(\delta H)_{xy} + (\delta M_2)_{yy} + \frac{\delta N_1}{R_1} + \frac{\delta N_2}{R_2} + N_1 w_{xx} + N_2 w_{yy} + 2T w_{xy} = 0, \quad (6)$$

where N_1, N_2, T are the membrane forces in the shell; R_1, R_2 are the principal radii of curvature; x, y are Cartesian coordinates;

$$\delta\chi_1 = w_{xx}, \qquad \delta\chi_2 = w_{yy}, \qquad \delta\chi_{12} = w_{xy}, \qquad \delta\varepsilon_1 = u_x - \frac{w}{R_1}, \qquad \delta\varepsilon_2 = \vartheta_y - \frac{w}{R_2},$$

$$\delta\gamma_{12} = u_y + \vartheta_x. \quad (7)$$

From (7) we obtain the compatibility condition

$$(\delta\varepsilon_1)_{yy} + (\delta\varepsilon_2)_{xx} - (\delta\gamma_{12})_{xy} + \frac{w_{yy}}{R_1} + \frac{w_{xx}}{R_2} = 0. \quad (8)$$

The first two equations are satisfied by the expressions $\delta N_1 = F_{yy}$, $\delta N_2 = F_{xx}$, $\delta T = -F_{xy}$.

We take w and F as the principal variables. In the third equation (6), (8), introduce expressions (4) and (5):

$$\alpha_1 w_{xxxx} - \alpha_2 w_{xxxy} + \alpha_3 w_{xxyy} - \alpha_4 w_{xyyy} + \alpha_5 w_{yyyy} - \alpha_6 F_{xxxx}$$

$$-\alpha_7 F_{xxxy} - \alpha_8 F_{xxyy} - \alpha_9 F_{xyyy} - \alpha_{10} F_{yyyy} - \frac{F_{yy}}{R_1} - \frac{F_{xx}}{R_2}$$

$$-\left(N_1 w_{xx} + N_2 w_{yy} + 2T w_{xy}\right) = 0; \quad (9)$$

$$\beta_1 F_{xxxx} + \beta_2 F_{xxxy} + \beta_3 F_{xxyy} + \beta_4 F_{xyyy} + \beta_5 F_{yyyy} + \beta_6 w_{xxxx}$$

$$+\beta_7 w_{xxxy} + \beta_8 w_{xxyy} + \beta_9 w_{xyyy} + \beta_{10} w_{yyyy} + \frac{w_{yy}}{R_1} + \frac{w_{xx}}{R_2} = 0. \qquad (10)$$

Here

$$\alpha_1 = D_{11}, \quad \alpha_2 = D_{13} + 2D_{31}, \quad \alpha_3 = D_{12} + D_{21} + 2D_{33}, \quad \alpha_4 = D_{23} + 2D_{32},$$
$$\alpha_5 = D_{22}, \quad \alpha_6 = C_{12}, \quad \alpha_7 = C_{13} - 2C_{32}, \quad \alpha_8 = C_{11} + C_{22} - 2C_{33},$$
$$\alpha_9 = C_{23} - 2C_{31}, \quad \alpha_{10} = C_{21}, \quad \beta_1 = A_{22}, \quad \beta_2 = A_{23} + A_{32},$$
$$\beta_3 = A_{12} + A_{21} + A_{33}, \quad \beta_4 = A_{13} + A_{31}, \quad \beta_5 = A_{11},$$
$$\beta_6 = A_{24}, \quad \beta_7 = A_{34} - A_{26}, \quad \beta_8 = A_{25} - A_{34} + A_{14}, \quad \beta_9 = A_{35} - A_{16},$$
$$\beta_{10} = A_{15}.$$

Let us note that for a homogeneous shell (1) $\alpha_6 = \alpha_7 = \alpha_8 = \alpha_9 = \alpha_{10} = 0$ and $\beta_6 = \beta_7 = \beta_8 = \beta_9 = \beta_{10} = 0$.

3. Critical loads for bimetallic structures

We shall consider the case of combined loads on the shell. Let the wave formation during buckling be of such a character that

$$w = w_0 \sin\left(x\sqrt{\lambda_1} - y\sqrt{\lambda_2}\right), \qquad F = F_0 \sin\left(x\sqrt{\lambda_1} - y\sqrt{\lambda_2}\right), \qquad (11)$$

where $\lambda_1 = m^2\pi^2/l^2$, $\lambda_2 = n^2/R_2^2$; m, n are the numbers of half-waves in the directions x and y; l is the shell length; w_0, F_0 are constants.

Substituting (11) into (9), (10), we find

$$-\left(N_1 + N_2\lambda - 2T\sqrt{\lambda}\right) = \lambda_1\left(\alpha_1 + \alpha_2\sqrt{\lambda} + \alpha_3\lambda + \alpha_4\lambda\sqrt{\lambda} + \alpha_5\lambda^2\right) +$$

$$+\frac{1}{\lambda\left(\beta_1 - \beta_2\sqrt{\lambda} + \beta_3\lambda - \beta_4\lambda\sqrt{\lambda} + \beta_5\lambda^2\right)}\left[\lambda_1\left(\beta_6 - \beta_7\sqrt{\lambda} + \beta_8\lambda - \beta_9\lambda\sqrt{\lambda} + \beta_{10}\lambda^2\right) -\right.$$

$$\left.-\left(\frac{\lambda}{R_1} + \frac{1}{R_2}\right)\right]\left[\lambda_1\left(\alpha_6 - \alpha_7\sqrt{\lambda} + \alpha_8\lambda - \alpha_9\lambda\sqrt{\lambda} + \alpha_{10}\lambda^2\right) - \left(\frac{\lambda}{R_1} + \frac{1}{R_2}\right)\right], \qquad (12)$$

where $\lambda = \lambda_2/\lambda_1 = n^2 l^2/m^2\pi^2 R_2^2$.

a) **Rectangular plate.** If a and b are the lengths of the sides of the plate, then $\lambda_1 = m^2\pi^2/a^2$, $\lambda_2 = n^2\pi^2/b^2$, $\lambda = n^2a^2/m^2b^2$. In addition, $R_1 = R_2 = \infty$. The corresponding changes must be introduced into the formula for determining the critical combination of forces (12). For an infinitely wide plate $\lambda = 0$ and $-N_1 = (\alpha + \beta_6\alpha_6/\beta_1)\lambda_1$, while the critical value of the compressive force is determined from $(m = 1)$

$$(-N_1)_* = \frac{\pi^2}{a^2}\left(\alpha_1 + \frac{\alpha_6\beta_6}{\beta_1}\right). \tag{13}$$

b) **Shells.** In the general case, the critical combination of forces is found from formula (12), if the corresponding values are substituted for the principal radii of curvature. If it may be assumed that $\lambda^2 \gg 1$, then from (12) we have

$$-\left(N_1 + N_2\lambda - 2T\sqrt{\lambda}\right) =$$

$$= \lambda_1\lambda^2\alpha_5 + \frac{1}{\beta_5\lambda_1\lambda^2}\left[\lambda_1\beta_{10}\lambda^2 - \left(\frac{\lambda}{R_1} + \frac{1}{R_2}\right)\right]\left[\lambda_1\lambda^2\alpha_{10} - \left(\frac{\lambda}{R_1} + \frac{1}{R_2}\right)\right]. \tag{14}$$

Let us consider various special cases. For a cylindrical shell $R_1 = \infty$ and $R_2 = R$; therefore from (14) we find

$$-\left(N_1 + N_2\lambda - 2T\sqrt{\lambda}\right) = \left(\alpha_5 + \frac{\alpha_{10}\beta_{10}}{\beta_5}\right)\lambda_1\lambda^2 - \frac{\alpha_{10} + \beta_{10}}{\beta_5 R} + \frac{1}{\beta_5 R^2\lambda_1\lambda^2}. \tag{15}$$

Hence, for example, under axial compression

$$-N_1 = \left(\alpha_5 + \frac{\alpha_{10}\beta_{10}}{\beta_5}\right)\lambda_1\lambda^2 - \frac{\alpha_{10} + \beta_{10}}{\beta_5 R} + \frac{1}{\beta_5 R^2\lambda_1\lambda^2},$$

when $\lambda_1\lambda^2 = 1/R\sqrt{\alpha_5\beta_5 + \alpha_{10}\beta_{10}}$, the critical value of the axial compressive force is

$$(-N_1)_* = \frac{1}{R\beta_5}\left[2\sqrt{\alpha_5\beta_5 + \alpha_{10}\beta_{10}} - (\alpha_{10} + \beta_{10})\right].$$

Analogous formulas can be obtained for a cylindrical shell under lateral pressure and torsion, for a spherical shell subjected to external pressure, etc.

The values b_{11}, $b_{12} = b_{21}$, b_{22}, b_{33} for various stress states were computed in paper (2).

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References

1. E. I. Grigolyuk, *Izv. AN SSSR, OTN*, No. 10, 3 (1957).

2. E. I. Grigolyuk, *Prikl. matem. i mekh.*, 21, No. 6 (1957).

Note: Figure translations are in progress. See original paper for figures.