MATHEMATICS

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Abstract

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ON THE INTEGRAL $\int_0^\pi \frac{\sin^m x}{p + q\cos x}\,dx$

(Presented by Academician A. N. Kolmogorov, 12 II 1958)

Let us denote

$$\Phi_m = \int_0^\pi \frac{\sin^m x}{p + q\cos x}\,dx; \tag{1}$$

$m \geqslant 2$ an integer; $q \neq 0$; $p^2 - q^2 \geqslant 0$.

The substitution $\cos^2 \frac{x}{2} = z$ brings (1) to the form:

$$\Phi_m = \frac{2^{m-1}}{q} \int_0^1 \frac{z^{\frac{m-1}{2}}(1-z)^{\frac{m-1}{2}}}{z+\beta}\,dz; \qquad \beta \equiv \frac{p-q}{2q}. \tag{2}$$

It is obvious that for $p^2 - q^2 > 0$ the integral (1) converges absolutely. For $p^2 - q^2 = 0$ the denominator of the integrand has one zero point at $x = 0$ or $x = \pi$. But at the same time $\sin x$ also vanishes. Thus, in the interval $0 \leq x \leq \pi$, for $m \geq 2$ the integrand has a pole if $p^2 - q^2 < 0$, and has none if $p^2 - q^2 = 0$.

In particular, for $p = q$, $\beta = 0$,

$$\Phi_m = \frac{2^{m-1}}{q} \int_0^1 z^{\frac{m-3}{2}}(1-z)^{\frac{m-1}{2}}\,dz =$$

$$= \frac{2^{m-1}}{q}\,\mathrm{B}\left(\frac{m-1}{2}, \frac{m+1}{2}\right) = \frac{2^{m-1}}{q}\,\frac{\Gamma\left(\frac{m-1}{2}\right)\Gamma\left(\frac{m+1}{2}\right)}{\Gamma(m)}. \tag{3}$$

The case $p = -q$ corresponds to $\beta = -1$ and to a change of sign in the right-hand side of result (3). Using in (1) the substitution $\pi - x = \alpha$, one can show that Φ_m is an even function with respect to the parameter q and an odd function with respect to p.

Thus, under the given conditions Φ_m has a finite and definite value. However, the handbook of I. M. Ryzhik, *Tables of Integrals, Sums, Series, and Products* (2nd ed., 1948), with reference to the table of integrals in the collection by D.

Bierens de Haan (Leiden, 1867),* gives for the integral (1) the result (see formula 11_{64} on p. 180 and I. M. Ryzhik' s note on p. 116, second footnote):

$$\Phi_m = -\frac{2\sqrt{\pi}}{m\,(p^2-q^2)^{\frac{m+1}{2}}}\,\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}. \tag{4}$$

* The integral under consideration is incorrectly evaluated also in the corrected American edition of 1939 (D. Bierens de Haan, *Nouvelles tables d' intégrales définies*, N. Y., 1939, p. 99, t. 63, No. 15).

As $p^2-q^2 \to +0$, $m \geq 2$, here $\Phi_m \to +\infty$. In addition, solution (4) is represented by an even function with respect to p. This contradicts the expectations following from a preliminary analysis of integral (1), and prompted us to doubt the correctness of formula (4) and of others based on it (for example, formula 13_{64} in the same place), which for 90 years have been included in mathematical handbooks.

Indeed:

$$\Phi_3 = \frac{4}{q}\int_0^1 \frac{z(1-z)}{z+\beta}\,dz = \frac{4}{q}\left[(1+\beta)z - \frac{1}{2}z^2 - \beta(1+\beta)\ln(z+\beta)\right]_0^1 =$$

$$= 2\,\frac{p}{q^2} + \frac{1}{q}\left(1-\frac{p^2}{q^2}\right)\ln\frac{p+q}{p-q}; \tag{5}$$

$$\Phi_2 = \frac{2}{q}\int_0^1 \frac{\sqrt{z(1-z)}}{z+\beta}\,dz = \frac{2p}{q^2}\left[\sqrt{1-\frac{q^2}{p^2}}\,\mathrm{arctg}\sqrt{\frac{1-z}{z}\cdot\frac{p-q}{p+q}}\right.$$

$$\left. -\,\mathrm{arctg}\sqrt{\frac{1-z}{z}} + \frac{q}{p}\sqrt{z(1-z)}\right]_0^1 = \pi\frac{p}{q^2}\left(1-\sqrt{1-\frac{q^2}{p^2}}\right), \tag{6}$$

and these results contradict solution (4) given in the handbooks.

Noting that

$$\Phi_{m+2} = \frac{2^{m+1}}{q}\int_0^1 \frac{z^{\frac{m-1}{2}}(1-z)^{\frac{m-1}{2}}}{z+\beta}[(z+\beta)-\beta][(1+\beta)-(z+\beta)]\,dz =$$

$$= \frac{2^{m+1}}{q}\left[\int_0^1 (1+\beta)z^{\frac{m-1}{2}}(1-z)^{\frac{m-1}{2}}\,dz\right.$$

$$-\int_0^1 z^{\frac{m+1}{2}}(1-z)^{\frac{m-1}{2}}\,dz-\beta(1+\beta)\int_0^1 \frac{z^{\frac{m-1}{2}}(1-z)^{\frac{m-1}{2}}}{z+\beta}\,dz\Bigg]=$$

$$=2^m\frac{p}{q^2}B\left(\frac{m+1}{2},\frac{m+1}{2}\right)+\left(1-\frac{p^2}{q^2}\right)\Phi_m, \tag{7}$$

we easily obtain the general solution of the problem under consideration:

$$\Phi_m=2^{m-2}\frac{p}{q^2}\sum_{\nu=1}^{k}\left(\frac{p^2-q^2}{-4q^2}\right)^{\nu-1}B\left(\frac{m+1-2\nu}{2},\frac{m+1-2\nu}{2}\right)+\left(\frac{p^2-q^2}{-q^2}\right)^{k}A,$$

where

$$A=\begin{cases}\dfrac{\pi p}{q^2}\left(1-\sqrt{1-\dfrac{q^2}{p^2}}\right), & \text{for } m=2k+2;\\[2ex] \dfrac{1}{q}\ln\dfrac{p+q}{p-q}, & \text{for } m=2k+1,\ k\geq 1.\end{cases}$$

For $m=2$ the solution is given by formula (6).

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Note: Figure translations are in progress. See original paper for figures.