HYDROMECHANICS

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Abstract

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APPROXIMATE ANALYTICAL REPRESENTATION OF PLANE SUPERSONIC GAS FLOWS

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Integration of the equations

1. The equations of motion describing supersonic flows of a compressible gas have the form (^{1}):

$$\frac{\partial^2\varphi}{\partial\lambda\,\partial\mu} = F(t)\left(\frac{\partial\varphi}{\partial\lambda} + \frac{\partial\varphi}{\partial\mu}\right), \qquad \frac{\partial^2\psi}{\partial\lambda\,\partial\mu} = -F(t)\left(\frac{\partial\psi}{\partial\lambda} + \frac{\partial\psi}{\partial\mu}\right), \tag{1,1}$$

where ψ and φ are, respectively, the stream function and the velocity potential; λ and μ are characteristic variables connected with the inclination angle of the velocity θ and the variable t as follows: $t - \theta = 2\lambda,\ t + \theta = 2\mu$.

The relation between the variable t and the velocity is determined from the system

$$\frac{d\rho}{dt} = -\frac{\rho^2 K + 1}{\sqrt{K}}, \qquad \frac{1}{\rho v} = -\sqrt{K}\,\frac{d}{dt}\left(\frac{1}{v}\right), \qquad F(t) = \frac{1}{2}\frac{d}{dt}(\ln\sqrt{K}), \tag{1,2}$$

where ρ is the ratio of the gas density to the stagnation density; v is the velocity magnitude.

In the case of adiabatic gas flow $t = \nu(M)$, where

$$\nu = \sqrt{\frac{\chi+1}{\chi-1}}\,\operatorname{arctg}\sqrt{\frac{\chi-1}{\chi+1}\,(M^2-1)} - \operatorname{arctg}\sqrt{M^2-1}, \tag{1,3}$$

$$F = \frac{\chi+1}{4}\,\frac{M^4}{(M^2-1)^{3/2}}. \tag{1,4}$$

Let us represent approximately

Fig. 1

Figure 1: Fig. 1

$$F = \frac{k}{c - \nu}, \tag{1,5}$$

where k and c are arbitrary constants.

The solution of system $(1,1)$ for integral k, given by Darboux, is represented in the form

$$\varphi = \frac{\partial^{2k-2}}{\partial \lambda^{k-1} \partial \mu^{k-1}} \left[\frac{f_1(\lambda) + f_2(\mu)}{\lambda + \mu - c} \right]. \tag{1,6}$$

Equating the derivatives with respect to ν of the inverse functions F, defined by relations $(1,4)$ and $(1,5)$, for $M \to \infty$, we obtain $k = (\chi + 1)/2(\chi - 1)$. For $\chi = 1.4$ and $\chi = 1.67$, k is equal to the integers 3 and 2, respectively.

S. A. Khristianovich $(^2)$ proposed approximating the solution of system $(1,1)$ by a solution of the type $(1,6)$ with $k = \pm 1$. Figure 1 shows the graph of the exact function F for $\chi = 1.4$ according to $(1,4)$, and points are plotted corresponding to the approximate values of F computed from $(1,5)$ for $k = 3$. The approximating curves chosen by S. A. Khristianovich $(^2)$ and G. A. Dombrovskii $(^3)$ are also indicated there. The best agreement with the exact values occurs in the case of approximation by $(1,5)$.

By choosing the constant c, the approximation curve can be drawn in such a way that it will have either one point of intersection with the exact curve, or a point of intersection and tangency as $M \to \infty$, or two points of intersection, or one point of tangency at $M = 4.57$, or no common point at all.

Fig. 1. 1—according to Khristianovich (2); 2, 3, 4—according to Dombrovskii (3);
5—exact dependence; points—the proposed approximation

2. We integrate the system (1,3), using the fact that $F(t) = 3/(c - t)$. We obtain

$$\frac{1}{\rho \bar{v}} = -(c_1 y_1 + c_2 y_2); \tag{2,1}$$

$$\frac{1}{\bar{v}} = -\frac{1}{A}(c_1 z_1 + c_2 z_2); \tag{2,2}$$

$$\sqrt{K} = \frac{A}{(c - t)^6}, \tag{2,3}$$

Fig. 2

Figure 2: Fig. 2

where $\bar{v}$ is the ratio of the velocity to the critical velocity; A, c_1, and c_2 are arbitrary constants; y_1, y_2, z_1, and z_2 are functions of the argument $\sigma = c - t$:

$$y_1 = \frac{1}{\sigma^3}\left[\left(1 - \frac{3}{\sigma^2}\right)\sin\sigma + \frac{3}{\sigma}\cos\sigma\right], \qquad y_2 = \frac{1}{\sigma^3}\left[\left(1 - \frac{3}{\sigma^2}\right)\cos\sigma - \frac{3}{\sigma}\sin\sigma\right],$$

$$z_1 = \sigma(15 - \sigma^2)\cos\sigma - (15 - 6\sigma^2)\sin\sigma, \qquad z_2 = -\sigma(15 - \sigma^2)\sin\sigma - (15 - 6\sigma^2)\cos\sigma.$$

Fig. 2. 1—$c = 2.28$, $A = 294$, $c_1 = 8.0$, $c_2 = 8.0$; 2—$c = 2.28$, $A = 294$, $c_1 = 8.5$, $c_2 = 8.0$; 3—$c = 2.28$; 4—$c = 2.278$

Using Bernoulli' s equation and relations (1,2), it is easy to obtain an expression for the pressure:

$$p = p_* - \rho_0 \int_{t_*}^{t} \frac{\bar{v}^2}{\sqrt{K}}\, dt. \tag{2,4}$$

Choosing the constants c, A, c_1, c_2 and p_*, one can obtain fourth-order tangency of the approximation curve $p(\rho)$ with the curve of adiabatic pressure variation.

One can draw a more general conclusion: a broad class of functions $F(t)$ chosen by us, with an arbitrary constant, will make it possible to construct a function $p(\rho)$ having fourth-order contact with the adiabatic dependence of p on ρ.

The general solution of system (1.1) will depend essentially on the form of the function $F(t)$, and, consequently, if the function $F(t)$ is chosen insufficiently close to the adiabatic one, the solution obtained will give the correct result only in a narrow range of variation of the density.

Figure 2 gives the relative deviations of the approximating function F from the adiabatic one for different c. There are also given the relative deviations of the functions $1/\vartheta$, computed from (2.1) and (2.2), from the corresponding functions of the adiabatic motion. The indicated approximation gives a very accurate result for all Mach numbers, beginning with $M = 2.5$.

Boundary-value problems

We shall carry out the solution of boundary-value problems for the case $x = 1.4$. In this case the solution has the form

$$\varphi(\lambda,\mu) = \frac{2}{(\lambda+\mu-c)^3}\left\{f_1''(\lambda) + f_2''(\mu) - \frac{6[f_1'(\lambda)+f_2'(\mu)]}{\lambda+\mu-c} + \frac{12[f_1(\lambda)+f_2(\mu)]}{(\lambda+\mu-c)^2}\right\}, \tag{3.0}$$

where $f_1(\lambda)$ and $f_2(\mu)$ are arbitrary functions determined from the boundary conditions.

3. The Goursat problem. Along the characteristics $\lambda = \lambda_0$ and $\mu = \mu_0$ there are prescribed, respectively, $\varphi = \varphi_1(\mu)$ and $\varphi = \varphi_2(\lambda)$. We shall assume that $\varphi_1(\mu_0) = \varphi_2(\lambda_0) = 0$. Introduce the notation

$$\lambda^* = \lambda + \mu_0 - c, \qquad \mu^* = \lambda_0 + \mu - c. \tag{3.1}$$

Then the solution of the Goursat problem can be written in the following form:

$$f_1(\lambda) = \frac{1}{2}\lambda^{*3}\int_{\lambda_0}^{\lambda}\int_{\lambda_0}^{\lambda}\varphi_2(\lambda)\,d\lambda\,d\lambda. \tag{3.2}$$

The turning of a given flow about a corner point is obtained as a special case of the Goursat problem if one of the functions φ_1 or φ_2 is set equal to zero.

4. The Cauchy problem. Along a known curve

$$\lambda = \lambda_*(\mu) \quad \text{or} \quad \mu = \mu_*(\lambda) \tag{4.1}$$

there are prescribed

$$\varphi = \varphi_1(\lambda) = \varphi_2(\mu), \qquad \psi = \psi_1(\lambda) = \psi_2(\mu). \tag{4.2}$$

Using the basic equations (1.1) and the solution (3.0), we obtain that

$$\begin{aligned}\frac{d^5 f_1}{d\lambda^5} &= 30\varphi_1(\lambda) + \frac{3}{2}\sigma_1\left[(\mu_*'^2 + 6\mu_*' + 15)\varphi_1'(\lambda) - 2\mu_*'^2\sqrt{K(\sigma_1)}\psi_1'(\lambda)\right] + \\ &+\frac{3}{4}\sigma_1^2\left\{2(\mu_*'+3)\varphi_1''(\lambda) + 2\mu_*'\sqrt{K(\sigma_1)}\psi_1''(\lambda) + \mu_*''\left[\varphi_1'(\lambda) + \sqrt{K(\sigma_1)}\psi_1'(\lambda)\right]\right\} + \\ &+\frac{\sigma_1^3}{4}\left[\varphi_1'''(\lambda) - \sqrt{K(\sigma_1)}\psi_1'''(\lambda)\right],\end{aligned} \tag{4.3}$$

where $\sigma_1 = \lambda + \mu_*(\lambda) - c$.

The formula for $d^5 f_2/d\mu^5$ is obtained from (4.3) by replacing in the right-hand side φ_1, ψ_1, λ, and σ_1, respectively, by φ_2, $-\psi_2$, μ, and $\sigma_2 = \lambda_*(\mu) + \mu - c$. In the case when the initial data are specified along the line $p =$ const or along a rectilinear wall, the functions $f_1(\lambda)$ and $f_2(\mu)$ are expressed through multiple integrals of the prescribed functions $\varphi_1(\lambda)$, $\varphi_2(\mu)$, $\psi_1(\lambda)$, and $\psi_2(\mu)$ in the first case and of $\varphi_1(\lambda)$ and $\varphi_2(\mu)$ in the second.

5. The problem with prescribed conditions on a characteristic and a free surface. Along the free surface $\sigma = \lambda + \mu - c$ assumes the constant value equal to σ_1. In addition, along it $\psi = 0$. Along the characteristic $\lambda = \lambda_0$ the function is prescribed:

$$\varphi = \varphi_1(\mu). \tag{5.1}$$

At the point of intersection of the characteristic with the free surface ($\lambda = \lambda_0,\ \mu = \mu_0$) we shall assume that $\varphi_1(\mu_0) = 0$.

The posed problem is solved with the aid of the functions

$$f_1(\lambda) = c_1(\lambda)e^{r\lambda} + e^{\alpha\lambda}\left[\cos(\beta\lambda)c_2(\lambda) + \sin(\beta\lambda)c_3(\lambda)\right],$$

$$f_2(\mu) = \frac{1}{2}(\lambda_0 + \mu - c)^3 \int_{\mu_0}^{\mu}\!\!\int \varphi_1(\mu)\,d\mu\,d\mu, \qquad (5,2)$$

where

$$c_1(\lambda) = A\beta \int_{\lambda_0}^{\lambda} e^{-r\lambda} F(\lambda)\,d\lambda,$$

$$c_2(\lambda) = -A \int_{\lambda_0}^{\lambda} e^{-\alpha\lambda}\left[(\alpha - r)\sin(\beta\lambda) + \beta\cos(\beta\lambda)\right] F(\lambda)\,d\lambda,$$

$$c_3(\lambda) = A \int_{\lambda_0}^{\lambda} e^{-\alpha\lambda}\left[(\alpha - r)\cos(\beta\lambda) - \beta\sin(\beta\lambda)\right] F(\lambda)\,d\lambda,$$

$$F(\lambda) = -f_2'''(\nu_0 - \lambda) + \frac{12}{\sigma_1} f_2''(\nu_0 - \lambda) - \frac{60}{\sigma_1^2} f_2'(\nu_0 - \lambda) + \frac{120}{\sigma_1^3} f_2(\nu_0 - \lambda),$$

$$A = \frac{\sigma_1^3}{12\sqrt{15}}, \qquad \nu_0 = \lambda_0 + \mu_0, \qquad r = \frac{12}{\sigma_1} - 2\alpha,$$

$$\alpha = \frac{1}{2\sigma_1}\left\{8 - \sqrt[3]{4}\left[\sqrt[3]{\sqrt{5}+1} - \sqrt[3]{\sqrt{5}-1}\right]\right\},$$

$$\beta = \frac{\sqrt{3}}{2\sigma_1}\sqrt[3]{4}\left[\sqrt[3]{\sqrt{5}+1} + \sqrt[3]{\sqrt{5}-1}\right].$$

6. Problem with prescribed conditions on a characteristic and a rectilinear wall

Along a rectilinear wall $\theta = \text{const}$. We shall assume that the wall is situated so that $\theta = 0$. Then along the wall $\lambda = \mu$. Along the characteristic $\lambda = \lambda_0$ the function is prescribed:

$$\varphi = \varphi_1(\mu). \tag{6,1}$$

Then the problem is solved with the aid of the functions

$$f_1(\lambda) = (2\lambda - c)^4 c_1(\lambda) + (2\lambda - c)^2 c_2(\lambda) + c_3(\lambda),$$

$$f_2(\mu) = \frac{1}{2}(\lambda_0 + \mu - c)^3 \int_{\mu_0}^{\mu}\!\!\int \varphi_1(\mu)\,d\mu\,d\mu, \tag{6,2}$$

where

$$c_1(\lambda) = \frac{1}{4}\int_{\lambda_0}^{\lambda}(2\lambda - c)^{-2}F(\lambda)\,d\lambda, \qquad c_2(\lambda) = -\frac{1}{2}\int_{\lambda_0}^{\lambda}F(\lambda)\,d\lambda,$$

$$c_3(\lambda) = \frac{1}{4}\int_{\lambda_0}^{\lambda}(2\lambda - c)^2 F(\lambda)\,d\lambda,$$

$$F(\lambda) = f_2'''(\lambda) - \frac{6}{2\lambda - c}f_2''(\lambda) + \frac{12}{(2\lambda - c)^2}f_2'(\lambda).$$

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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