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M. L. LIDOV

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Abstract

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M. L. LIDOV

ON LIMITING SOLUTIONS NEAR A SINGULAR POINT

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In constructing a solution linearized near a strong explosion ([1]) for the axisymmetric problem of an explosion in a medium with variable density, V. P. Karlikov ([2]) discovered that the linearized solution cannot satisfy the boundary condition at the center of the explosion. This solution has a singularity at the center, a consequence of which is the appearance of infinite values of the velocity in the linearized solution. An analogous phenomenon was also found in the one-dimensional problem when linearizing near a particular solution considered in the work. Below, using a specific example, we investigate the solution of nonlinear equations near a singular point and present considerations on the construction of limiting solutions in the general case.

1. We shall start from the equations of one-dimensional unsteady adiabatic motions of an ideal gas (([1]), p. 169). For the adiabatic exponent $\gamma = 7$, the self-similar problem of a strong (the initial pressure is zero) point explosion in a medium of constant density ρ_1 has the simple solution (([1]), p. 229)

$$u_a = \frac{1}{10}\frac{r}{t}, \qquad p_a = \frac{1}{25}\frac{r^5}{t^{18/5}}\left(\frac{\rho_1}{E}\right)^{1/5}\rho_1, \qquad \rho_a = \frac{4}{3}\rho_1\left(\frac{\rho_1}{E}\right)^{1/5}\frac{r}{t^{2/5}}. \tag{1}$$

Here u_a, p_a, ρ_a are the velocity, density, and pressure; E is a constant proportional to the released energy; r is the distance from the explosion point; t is time. This solution satisfies the boundary conditions at the shock wave and at the center of the explosion (the velocity at the center is zero).

By the method of linearizing the full nonlinear system of partial differential equations near the solution (1), one can consider the non-self-similar problem of a strong point explosion in a medium with the initial density distribution

$$\rho_{\text{n}} = \rho_1[1-(\varepsilon r)^k], \qquad \varepsilon, k \text{ are constants.} \tag{2}$$

The solution of the original equations is sought in the form

$$u = u_a(1 + \gamma^k f(\lambda) + O(\gamma^{2k})), \qquad p = p_a(1 + \gamma^k h(\lambda) + O(\gamma^{2k})),$$

$$\rho = \rho_a(1 + \gamma^k g(\lambda) + O(\gamma^{2k})), \tag{3}$$

where

$$\lambda = \left(\frac{\rho_1}{E}\right)^{1/5} \frac{r}{t^{2/5}}, \qquad \gamma = \varepsilon \left(\frac{E}{\rho_1}\right)^{1/5} t^{2/5}$$

are dimensionless arguments; $f(\lambda), h(\lambda), g(\lambda)$ are dimensionless functions characterizing the deviation of the solution from the self-similar one.

After linearization with respect to γ^k, for f, h, and g we obtain a system of ordinary equations with constant coefficients, whose general solution has the form

$$f(\lambda) = \frac{3n_2 - 4k}{n_2 + 4} c_2 \lambda^{n_2} + \frac{3n_3 - 4k}{n_3 + 4} c_3 \lambda^{n_3}, \qquad g(\lambda) = c_1 \lambda^{n_1} + c_2 \lambda^{n_2} + c_3 \lambda^{n_3},$$

$$h(\lambda) = \frac{3}{n_1 + 3} c_1 \lambda^{n_1} + \frac{7n_2 + 24}{n_2 + 4} c_2 \lambda^{n_2} + c_3 \frac{7n_3 + 24}{n_3 + 4} \lambda^{n_3}. \tag{4}$$

where c_1, c_2, c_3 are arbitrary constants,

$$n_1 = \frac{4}{3}k, \qquad n_{2,3} = \frac{-(12k + 51) \pm \sqrt{336k^2 + 840k + 441}}{12}. \tag{5}$$

After the corresponding linearization, the law of motion of the shock wave $r_2(t)$ and the boundary conditions on the shock wave for the sought functions are given by the formulas

$$r_2(t) = \left(\frac{E}{\rho_1}\right)^{1/5} t^{2/5} \left(1 + \nu^k \beta + O(\nu^{2k})\right),$$

$$f(1) = k\beta, \qquad g(1) = -\beta - 1, \qquad h(1) = 2\beta k - 1 - \beta. \tag{6}$$

Three cases are considered: a) $k > 4.5$, b) $\frac{1 + \sqrt{91}}{4} < k < 4.5$, c) $0 < k < \frac{1 + \sqrt{91}}{4}$. In case a) it is possible to satisfy conditions (6) and the condition that the velocity be zero at the center of symmetry by setting $c_3 = 0$ and determining c_1, c_2, β from relations (6). In case b) $n_1 > 0$, $-1 < n_2 < 0$, $n_3 \ll -1$; here, taking into account the dimensional factor in (3), it is possible to satisfy all conditions by setting $c_3 = 0$; however, for sufficiently small λ, $\nu^k f(\lambda) > 1$, and the closeness of the linearized solution to the exact one is doubtful. In case c) $n_1 > 0$, $n_2 < -1$, $n_2 \ll -1$, and it is impossible for the linearized solution to satisfy simultaneously conditions (6) and the condition at the center of symmetry.

To obtain a solution near $\lambda = 0$ in cases b) and c), we introduce into the original nonlinear equations the dimensionless functions V, P, and R:

$$u = u_{aV}(\lambda, \nu), \qquad p = p_{aP}(\lambda, \nu), \qquad \rho = \rho_{aP}(\lambda, \nu). \tag{7}$$

For V, P, and R we obtain the corresponding equations. We shall determine V, P, and R near the center ($\lambda = 0$) from the following conditions:

1) When $\nu^k = 0$, the solution (7) coincides with the self-similar one:

$$V = P = R = 1. \tag{8}$$

2) The linearization of V, P, and R with respect to ν^k coincides with the behavior of the linearized solution (4) near the center. For the case when $c_3 = 0$ in (4), this condition has the form

$$\left.\frac{\partial V}{\partial(\nu^k)}\right|_{\nu=0} = \frac{3n_2 - 4k}{n_2 + 4} c_2 \lambda^{n_2}, \qquad \left.\frac{\partial P}{\partial(\nu^k)}\right|_{\nu=0} = \frac{7n_2 + 24}{n_2 + 4} c_2 \lambda^{n_2}, \qquad \left.\frac{\partial R}{\partial(\nu^k)}\right|_{\nu=0} = c_2 \lambda^{n_2}. \tag{9}$$

The system of nonlinear partial differential equations for V, P, and R and conditions (8) and (9) are invariant with respect to transformations of the independent variables of the form

$$\lambda_1^{n_2} = D\lambda^{n_2}, \qquad \nu_1^k = \nu^k / D. \tag{10}$$

Consequently, near $\lambda = 0$ the solution is determined by functions of one variable $\xi = \nu^k \lambda^{n_2}$, and the solution of the problem reduces to the integration of a system of ordinary nonlinear equations.

We carried out a qualitative investigation of these equations for $k = 1$ (case c)) and $k = 3$ (case b)). In the case $k = 1$, the investigation showed that near the center of symmetry a gas-free region is formed. The boundary of the region expands according to the law

$$r^* = \frac{1}{\xi_0^{1/|n_2|}} \left(\frac{E}{\rho_1}\right)^{1/5} t^{2/5} \nu^{1/|n_2|} \quad (\xi_0 = \text{const}).$$

At $r = r^*$, the conditions at the contact discontinuity are satisfied: the pressure and density are zero, and the velocity is equal to dr^*/dt. For $k = 3$, the solution can be continued to the center. Near the center $u = u_a(10 + c_1 \xi^{-10/9|n_2|})$, $c_1 = \text{const}$; the pressure and density also tend to zero, and more rapidly than in the self-similar case. The formation of a cavity near the center, revealed by the limiting solu-

solution, naturally owing to the specifics of solution (1), investigated in the book (1), p. 229.

2. By analogy with the example considered, in order to obtain a solution near the center, a method is proposed for constructing limiting solutions in the more general case of spatial perturbations of one-dimensional self-similar motion. In doing so, for definiteness, we shall have in mind axisymmetric motions close to one-dimensional spherically symmetric self-similar motions, considered in spherical coordinates.

Let A and B be constants associated with the self-similar motion, and let ε be a constant characterizing the perturbation. Without loss of generality ([1]), we assume that these parameters have dimensions $[A] = ML^{\chi}T^{\zeta}$, $[B] = L^{\mu}T^{\delta}$, $[\varepsilon] = 1/L$. Introduce dimensionless functions and dimensionless arguments

$$v_r = B^{\frac{1}{\mu}} t^{-\frac{\delta}{\mu}-1} V_r(\lambda, \nu, \theta), \qquad v_\theta = B^{\frac{1}{\mu}} t^{-\frac{\delta}{\mu}-1} V_\theta(\lambda, \nu, \theta),$$

$$p = AB^{-\frac{\chi+1}{\mu}} t^{-\zeta + \frac{\delta}{\mu}(\chi+1) - 2} P(\lambda, \nu, \theta), \qquad \rho = AB^{-\frac{\chi+3}{\mu}} t^{-\zeta + \frac{\delta}{\mu}(\chi+3)} R(\lambda, \nu, \theta),$$

$$\lambda = B^{-\frac{1}{\mu}} r t^{\frac{\delta}{\mu}}, \qquad \nu = \varepsilon B^{\frac{1}{\mu}} t^{-\frac{\delta}{\mu}}.$$

Here v_r and v_θ are the radial and transverse velocities; θ is the angular coordinate.

Let

$$\text{for } \nu = 0 \qquad V_\theta = 0, \quad V_r = V_r^a(\lambda), \quad R = R^a(\lambda), \quad P = P^a(\lambda), \tag{11}$$

where V_r^a, R^a, P^a are the dimensionless solution of the spherically symmetric self-similar problem. For V_r^a, R^a, and P^a near the center, the representation ([1]) in the form of series holds

$$V_r^a = \lambda \sum_{j=0}^{\infty} \lambda^{mj} \beta_j, \qquad P^a = \gamma_{-1} + \lambda^m \sum_{j=0}^{\infty} \lambda^{mj} \gamma_j, \qquad R = \lambda^s \sum_{j=0}^{\infty} \lambda^{mj} \alpha_j,$$

where β_j, γ_j, α_j, s are constants known from the self-similar solution; $m = s+2$.

We shall seek the dimensionless solution of the axisymmetric equations of non-stationary adiabatic motions of an ideal perfect gas in the form of series

$$V_r(\lambda,\nu,\theta) = \lambda \sum_{j=0}^{\infty} \lambda^{mj} V_r^j(\lambda,\nu,\theta), \qquad V_\theta(\lambda,\nu,\theta) = \lambda \sum_{j=0}^{\infty} \lambda^{mj} V_\theta^j(\lambda,\nu,\theta),$$

$$P(\lambda,\nu,\theta) = p_0(\nu) + \lambda^m \sum_{j=0}^{\infty} \lambda^{mj} P^j(\lambda,\nu,\theta), \qquad R = \lambda^s \sum_{j=0}^{\infty} \lambda^{mj} R^j(\lambda,\nu,\theta). \quad (12)$$

Substituting (12) into the corresponding equations, one can formally satisfy these equations if V_r^j, V_θ^j, R^j, P^j are connected by an infinite series of systems of equations. Without writing out the general form of these systems because of the cumbersome formulas, let us write the system of equations relating the first coefficients:

$$cp_0(\nu) + a\nu \frac{\partial p_0(\nu)}{\partial \nu} + \gamma p_0(\nu)\left(3V_r^0 + \lambda \frac{\partial V_r^0}{\partial \lambda}\right) + \frac{\gamma p_0}{\sin\theta} \frac{\partial V_\theta^0 \sin\theta}{\partial \theta} = 0,$$

$$(V_r^0 - a)\left(sR^0 + \lambda \frac{\partial R^0}{\partial \lambda}\right) + bR^0 + a\nu \frac{\partial R^0}{\partial \nu} + V_\theta^0 \frac{\partial R^0}{\partial \theta}$$
$$+R^0\left(3V_r^0 + \lambda \frac{\partial V_r^0}{\partial \lambda}\right) + \frac{R^0}{\sin\theta} \frac{\partial V_\theta^0 \sin\theta}{\partial \theta} = 0,$$

$$(V_r^0 - a)\left(V_\theta^0 + \lambda \frac{\partial V_\theta^0}{\partial \lambda}\right) + (a-1)V_\theta^0 + a\nu \frac{\partial V_\theta^0}{\partial \nu} +$$
$$+\frac{1}{R^0}\frac{\partial P^0}{\partial \theta} + V_\theta^0 \frac{\partial V_\theta^0}{\partial \theta} + V_\theta^0 V_r^0 = 0, \quad (13)$$

$$(V_r^0 - a)\left(V_r^0 + \lambda \frac{\partial V_r^0}{\partial \lambda}\right) + (a-1)V_r^0 + a\nu \frac{\partial V_r^0}{\partial \nu} +$$
$$+\frac{(s+2)P^0}{R^0} + \frac{\lambda}{R^0}\frac{\partial P^0}{\partial \lambda} + V_\theta^0 \frac{\partial V_r^0}{\partial \theta} - (V_\theta^0)^2 = 0,$$

where a, b, c are constants depending on ε, χ, δ, μ; γ is the adiabatic exponent.

Condition (11) requires that

$$\text{for } \nu = 0 \qquad V_r^0 = \beta_0, \quad p_0 = \gamma_{-1}, \quad V_\theta^0 = 0, \quad P^0 = \gamma_0, \quad R^0 = \alpha_0. \quad (14)$$

Suppose we know the solution of the axisymmetric problem linearized in ν^k. It can be shown that, for the functions V_r^0, V_θ^0, P^0, R^0, and p_0, in a number of interesting cases, as $\lambda \to 0$, it is representable in the form

$$V_r^0 = \beta_0 + \nu^k \lambda^\alpha f_1(\theta), \qquad V_\theta^0 = \nu^k \lambda^\alpha f_2(\theta),$$

$$P^0 = \gamma_0 + \nu^k \lambda^\alpha f_3(\theta), \qquad R^0 = \alpha_0 + \nu^k \lambda^\alpha f_4(\theta), \qquad p_0 = \gamma_{-1} + c\nu^k, \quad (15)$$

where $f_i(\theta)$ are known functions; c, α are constants. The cases $\alpha < 0$ are investigated.

We assume that the behavior of the characteristics of the motion as $\lambda \to 0$ and $\nu \to 0$ is determined by the solution of system (13) under conditions (14) and the second condition: coincidence of the linearization of the solution found from (13) with the solution of the linearized equations (15). This condition has the form

$$\text{for } \nu = 0 \qquad \frac{\partial V_r^0}{\partial(\nu^k)} = \lambda^\alpha f_1(\theta), \qquad \frac{\partial V_\theta^0}{\partial(\nu^k)} = \lambda^\alpha f_2(\theta),$$

$$\frac{\partial P^0}{\partial(\nu^k)} = \lambda^\alpha f_3(\theta), \qquad \frac{\partial R^0}{\partial(\nu^k)} = \lambda^\alpha f_4(\theta). \quad (16)$$

Equations (13) and conditions (14) and (16), as $\lambda \to 0$, $\nu \to 0$, are invariant with respect to the transformation of the independent variables $\lambda_1^\alpha = D\lambda^\alpha$, $\nu_1^k = \nu^k/D$. Consequently, the solution as $\lambda \to 0$ and $\nu \to 0$ will be determined by functions of two variables θ and $\xi = \nu^k \lambda^\alpha$, satisfying the corresponding partial differential equations in θ and ξ; moreover, according to (14) and (16), for this system at $\xi = 0$ the values of the functions and of the first derivatives with respect to ξ are prescribed, and the value $\xi = \infty$ corresponds to the center of the explosion.

A similar simplification of the equations and boundary conditions near the center can also be carried out in the investigation of perturbations about one-dimensional self-similar motions with plane and cylindrical symmetry, and in the general case of spatial perturbations.

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CITED LITERATURE

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