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**Abstract**

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**MATHEMATICS**

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# ON MEROMORPHIC FUNCTIONS TAKING CERTAIN VALUES AT POINTS LYING NEAR A FINITE SYSTEM OF RAYS

*(Presented by Academician S. N. Bernstein on 6 II 1958)*

In papers ($^{1-4}$) there were considered functions $f(z)$, meromorphic for $|z| < \infty$, which possess values attained in a certain neighborhood of a finite system of rays

$$\arg z = \theta_n, \quad n = 1, 2, \dots, m, \quad 0 \le \theta_1 < \theta_2 < \dots < \theta_m < 2\pi. \tag{1}$$

It then turned out that, if any one of these values is assumed by the function $f(z)$ sufficiently rarely (in the sense that, at least, its Nevanlinna defect is positive), then this entails an estimate of the growth of $T(r, f)$ depending only on the disposition of the rays (1).

A result of A. Edrei ($^3$) is as follows:
Let the roots of the three equations

$$f(z) = 0, \quad f^{-1}(z) = 0, \quad f^{(l)}(z) = 1 \tag{2}$$

($l$ is some nonnegative integer), with the possible exception of a finite number, lie on the rays (1). In addition, let

$$\delta(0, f) + \delta(\infty, f) + \delta(1, f^{(l)}) > 0. \tag{3}$$

Then the order of $f(z)$ does not exceed $\pi\gamma^{-1}$, where $\gamma = \min_{1 \le n \le m}(\theta_{n+1} - \theta_n)$ ($\theta_{m+1} = 2\pi + \theta_1$).

In paper ($^4$), which generalizes results of ($^2$), an assertion is proved that may be formulated as follows:
Let the function $f(z)$ be representable in the form

$$f(z)=\sum A_k(z-h_k)^{-1},\quad \sum|A_{kh}k^{-1}|<\infty,\quad \sum|\operatorname{Im}(h_k^{-1})|<\infty,\quad (4)$$

and let $\delta(b,f)>0$ for some $b$. Then

$$\lim_{r\to\infty} r^{-1}T(r,f)<\infty.$$

The connection between this assertion and the result ([3]) becomes immediately apparent if one uses the following simple observation. If $f(z)$ is represented in the form (4), then for any $a$ (including 0 and $\infty$) the relation

$$\sum|\operatorname{Im} z_k^{-1}(a)|<\infty$$

holds, where $z_k(a)$ are the roots of the equation $f(z)=a$. Consequently one may say that the roots of the equations

$$f(z)=0,\quad f^{-1}(z)=0,\quad f(z)=1$$

are situated "near" the rays $\arg z=0$ and $\arg z=\pi$. Moreover, since, without loss of generality, one may take $b=1$, condition (3) is satisfied with $l=0$. Meanwhile, Edrei' s theorem is not applicable here.

Comparison of the results ([3]) and ([4]) led the author to the establishment of a theorem with respect to which both results are particular cases. Before formulating it, we introduce the necessary notation.

Let $f(z)$ be a function meromorphic for $|z|<\infty$, and let $r_k e^{i\varphi_k}$ be its poles. Following (1), put

$$C(R,\alpha,\beta,f)=2\sum_{\substack{1\le r_k<R\\ \alpha<\varphi_k<\beta}}\left(\frac{1}{r_k^{\pi/\gamma}}-\frac{r_k^{\pi/\gamma}}{R^{2\pi/\gamma}}\right)\sin\frac{\pi}{\gamma}(\varphi_k-\alpha)$$

$$(0<\beta-\alpha=\gamma\le 2\pi).$$

By the symbols $K_1(t)$ and $K_2(t)$ we shall denote positive nondecreasing functions of $t\ge 0$, and by the letters $k_i$ the quantities

$$\overline{\lim}_{t\to\infty}\ln K_i(t)(\ln t)^{-1},\quad i=1,2.$$

**Definition 1.** The set of $a$-points of the function $f(z)$ is called "close" to the system of rays (1) if the inequality

$$\sum_{n=1}^{m} C(R,\theta_n,\theta_{n+1},(f-a)^{-1})\le K_1(R)K_2(T(R,f)),$$

where $k_1$ is finite and $k_2 < 1$, holds for all $R \geq 0$, except perhaps for some set $\mathfrak{A} \subset [0, \infty)$ such that

$$\lim_{R\to\infty} R^{-1} \operatorname{mes}\{\mathfrak{A} \cap [0, R]\} = 0.$$

**Definition 2.** A number $a$ is called a $*$-deficient value of the function $f(z)$ if there exists a set $\mathfrak{B} \subset [0, \infty)$ such that:

1) $\overline{\lim}_{R\to\infty} R^{-1} \operatorname{mes}\{\mathfrak{B} \cap [0, R]\} < 1$; 2) $\lim\limits_{\substack{R\to\infty \\ R\in\mathfrak{B}}} m(R, a, f)(T(R, f))^{-1} > 0$.

Obviously, a Nevanlinna deficient value is *a fortiori* $*$-deficient.

**Theorem 1.** Let $f(z)$ be a function meromorphic for $|z| < \infty$, and suppose that at least one of the conditions A, B, C is satisfied:

A. 1) The zeros and poles of $f(z)$ are "close" to the system of rays (1); 2) at least one of the functions $f^{(l)}(z)$, $l \geq 0$, has at least one $*$-deficient value distinct from 0 and $\infty$.

B. 1) The poles of $f(z)$ and the $a$-points of $f^{(l)}(z)$, for some $a \neq 0, \infty$ and some integer $l \geq 0$, are "close" to the system of rays (1); 2) zero is a $*$-deficient value of $f(z)$.

C. 1) The zeros and poles of $f(z)$ and the $a$-points of $f^{(l)}(z)$, for some $a \neq 0, \infty$ and some integer $l \geq 0$, are "close" to the system of rays (1); 2) $\infty$ is a $*$-deficient value of $f(z)$.

Then the order of the function $f(z)$ is finite and does not exceed the quantity

$$\chi = \chi(\gamma, k_1, k_2) = (\pi + \gamma k_1)\gamma^{-1}(1 - k_2)^{-1},$$

where

$$\gamma = \min_{1\leq n\leq m} (\theta_{n+1} - \theta_n), \quad \theta_{m+1} = 2\pi + \theta_1.$$

Moreover, if both quantities

$$\sigma_i = \overline{\lim}_{t\to\infty} K_i(t) t^{-k_i} \quad (i = 1, 2)$$

are finite, then the growth of $T(R, f)$ does not exceed the normal type of order $\chi$, and if in addition one of them is equal to zero, then it does not exceed the minimal type of order $\chi$.*

For $k_1 = k_2 = 0$ we obtain that the order of $f(z)$ does not exceed $\pi\gamma^{-1}$, and if, moreover, $K_i(t) = O(1)$ $(i = 1, 2)$ (which is always fulfilled if, apart from a finite number, the roots of equations (2) are situated on the rays (1)), then the type is not above normal. This assertion is stronger than Edrei' s theorem ([3]) and stronger than the result ([4]).

For the proof of Theorem 1 the following are used: a) the estimate

$$m(R, a, f^{(l)}) \le m(R, f^{(l)}/f^{(l+1)}) + C_l \ln\{RT(R, f)\}; \quad l = 0, 1, \dots; \quad a \ne 0, \infty,$$

valid

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* One can also estimate the refined order of $T(R, f)$ through the refined orders $K_1(t)$ and $K_2(t)$.

for all $R \ge 0$, except perhaps for a set of finite length; b) an estimate of the quantity $m(R, f^{(l)}/f^{(l+1)})$, following from the following theorem, which lies at the basis of our investigation:

**Theorem 2.** Whatever may be given: a function $f(z)$ meromorphic for $|z| < \infty$; $l = 0, 1, 2, \dots$; $\alpha$ and $\beta$ $(0 < \beta - \alpha = \gamma \le 2\pi)$; $\varepsilon > 0$, there exists a set $\mathfrak{A} = \mathfrak{A}_{f,l,\alpha,\beta,\varepsilon} \subset [0, \infty)$ such that:

1)
$$\overline{\lim_{R\to\infty}} R^{-1} \operatorname{mes}\{\mathfrak{A} \cap [0, R]\} < \varepsilon;$$

2)
$$\int_\alpha^\beta \ln^+ \left| \frac{f^{(l)}(Re^{i\theta})}{f^{(l+1)}(Re^{i\theta})} \right| d\theta \le A_{f,l,\alpha,\beta,\varepsilon} \ln^4\{RT(R, f)\} \times$$
$$\times R^{4\pi/\gamma}\{C(\widetilde{R}, \alpha, \beta, f) + C(\widetilde{R}, \alpha, \beta, f^{-1}) + q(\gamma, R)\ln\{RT(R, f)\}\} \quad (5)$$

for $R \in C\mathfrak{A}$, where

$$\widetilde{R} = R + \frac{R}{\ln T(R, f)}; \qquad q(\gamma, R) = \begin{cases} 1, & (0 < \gamma \le \pi), \\ R^{1-\pi/\gamma}, & (\pi < \gamma \le 2\pi). \end{cases}$$

If the order of $f(z)$ is finite, then inequality (5) may be replaced by the more precise one:

2′)
$$\int_\alpha^\beta \ln^+ \left| \frac{f^{(l)}(Re^{i\theta})}{f^{(l+1)}(Re^{i\theta})} \right| \le A_{f,l,\alpha,\beta,\varepsilon} R^{\pi/\gamma}\{C(4^{\gamma/\pi}R, \alpha, \beta, f) +$$
$$+ C(4^{\gamma/\pi}R, \alpha, \beta, f^{-1}) + 1\}.$$

The known theorem of Nevanlinna on the logarithmic derivative makes it possible to estimate from above the quantity $m(r, f')$ in terms of $m(r, f)$. Theorem 2 may be applied in some cases to obtain the converse estimate.

In the proof we start from Nevanlinna' s formula ([1]) and proceed by methods that constitute a refinement of the methods of ([4]). Our methods also make it possible to obtain the following assertion:

**Theorem 3.** Let $f(z)$ be meromorphic in the disk $|z| < 1$, and suppose that at least one of the conditions A, B, C is satisfied:

A. 1) The zeros and poles of $f(z)$ lie on the radii (1). 2) $\delta(a, f^{(l)}) > 0$ for some $a \neq 0, \infty$ and some integer $l \geq 0$.

B. 1) The poles of $f(z)$ and the $a$-points of $f^{(l)}(z)$, for some $a \neq 0, \infty$ and some integer $l \geq 0$, lie on the radii (1); 2) $\delta(0, f) > 0$.

C. 1) The zeros and poles of $f(z)$ and the $a$-points of $f^{(l)}(z)$, for some $a \neq 0, \infty$ and some integer $l \geq 0$, lie on the radii (1); 2) $\delta(\infty, f) > 0$.

Then, independently of the number (provided only that it is finite) and the arrangement of the radii (1), the order of $f(z)$, i.e.

$$\lim_{r \to 1} \frac{\ln T(r, f)}{\ln(1/(1-r))}$$

does not exceed 4.

We note that there is also a stronger theorem, in whose hypothesis there occur (adapted to the hyperbolic case) the notions of "proximity" of the set of $a$-points of $f(z)$ to the radii (1) and of a $*$-defective value.

I express my gratitude to A. A. Goldberg, to whom I owe the idea of comparing the results ([3]) and ([4]), and to B. Ya. Levin for a number of valuable suggestions and attention to the work.

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*Note: Figure translations are in progress. See original paper for figures.*