G. F. MANDZHAVIDZE and B. V. KHVEDELIDZE

1958

SovietRxiv

Abstract

Full Text

G. F. MANDZHAVIDZE and B. V. KHVEDELIDZE

ON THE RIEMANN-PRIVALOV PROBLEM WITH CONTINUOUS COEFFICIENTS

(Presented by Academician N. I. Muskhelishvili on 11 VII 1958)

1. Let Γ be a simple closed Lyapunov curve. Denote by E^+ the finite domain, and by E^- the infinite domain, bounded by the line Γ. In what follows, functions of a complex variable defined in the domains E^+ and E^- will be marked, respectively, by plus and minus signs.

A function of the form

$$\Phi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(t)}{t-z}\, dt + P(z),$$

where $\varphi(t)$ is a summable function on Γ, and $P(z)$ is an entire function, will be called an integral of Cauchy type with principal part $P(z)$ at infinity.

In what follows, when we say that a function of a complex variable belongs to the class $L_p(\Gamma)$ $(p > 1)$, this will mean that the angular boundary values of this function on Γ belong to $L_p(\Gamma)$.

We note that if the Cauchy-type integrals $\Phi(z)$ and $\Psi(z)$ with principal parts at infinity belong respectively to the classes $L_p(\Gamma)$ and $L_q(\Gamma)$, $p > 1$, $q = p(p-1)^{-1}$, then the product $\Phi(z)\Psi(z)$ will also be a Cauchy-type integral with some principal part at infinity (^{1}).

Let $a(t)$ be a measurable bounded matrix on Γ, i.e. the elements of the matrix $a(t)$ are measurable bounded functions defined on Γ. A matrix $X(z) \in L_p(\Gamma)$, $p > 1$, will be called a canonical matrix for the matrix $a(t)$ if $X^{-1}(z) \in L_q(\Gamma)$, $q = p(p-1)^{-1}$; $X(z)$, $X^{-1}(z)$ are representable by Cauchy integrals with polynomial principal parts at infinity; $X(z)$ has normal form at infinity and almost everywhere on Γ

$$a(t) = X^+(t)[X^-(t)]^{-1}. \tag{1}$$

If for the matrix $a(t)$ there exists a canonical matrix, then we shall say that $a(t)$ is representable by means of a canonical matrix.

2. Consider the following boundary-value problem: to find a matrix $\Phi(z) \in L_p(\Gamma)$, $p > 1$, whose elements are representable by Cauchy-type integrals with some principal parts at infinity, satisfying the boundary condition

$$\Phi^+(t) = a(t)\Phi^-(t) + b(t), \qquad t \in \Gamma, \tag{2}$$

which holds almost everywhere, where $a(t)$ is a given bounded measurable square matrix of order n, representable by means of a canonical matrix $X(z)$; $b(t)$ is a given matrix satisfying the condition $[X^+(t)]^{-1}b(t) \in L_p(\Gamma)$.

It is easy to show (cf. (1)) that all solutions of problem (2) are represented by the formula

$$\Phi(z) = \frac{X(z)}{2\pi i} \int_\Gamma \frac{[X^+(t)]^{-1}b(t)}{t-z}\, dt + X(z)P(z), \tag{3}$$

where $P(z) = (P_{ik})$, the $P_{ik}(z)$ being arbitrary entire functions.

Thus, the entire difficulty of solving problem (2) consists in constructing, or proving the existence of, a canonical matrix for $a(t)$.

It is known ([2],[3]) that if a nonsingular matrix $a(t)$ satisfies a Hölder condition, then it is representable by means of a canonical matrix. Below we shall show that if a nonsingular matrix $a(t)$ is continuous on Γ, then it is representable by means of a canonical matrix. Consequently, under this assumption formula (3) gives all solutions of problem (2).

3. Suppose that $a(t)$ is a given nonsingular continuous matrix of order n; $b(t)$ is a matrix of the same order, belonging to the class $L_p(\Gamma)$, $p > 1$. We shall prove that under these assumptions there exist matrices $\Phi^+(z)$, $\Phi^-(z)$, satisfying almost everywhere on Γ condition (2), where $\Phi^+(z)$ is representable by an integral of Cauchy type, while $\Phi^-(z)$ is representable by an integral of Cauchy type after multiplication on the left by a certain rational matrix.*

Since $a(t)$ is a nonsingular continuous matrix, for any positive ε there exists a rational matrix $r(z)$, nonsingular on Γ, whose elements have no poles on Γ, such that

$$|a(t)r^{-1}(t) - I| < \varepsilon, \qquad |a^{-1}(t)r(t) - I| < \varepsilon, \qquad t \in \Gamma, \tag{4}$$

where I is the identity matrix, and the conditions (4) mean that the corresponding inequalities are satisfied by all elements of the matrices enclosed in the modulus signs.

Now, subtracting from both sides of equality (2) the matrix $r(t)\Phi^-(t)$ and introducing the notation $\varphi^+(z) = \Phi^+(z)$, $\varphi^-(z) = r(z)\Phi^-(z)$, we obtain

$$\varphi^+(t) - \varphi^-(t) = a_1(t)\varphi^-(t) + b(t), \tag{5}$$

where $a_1(t) = a(t)r^{-1}(t) - I$.

Thus, the problem has been reduced to finding matrices $\varphi^+(z)$, $\varphi^-(z)$, which are representable by Cauchy-type integrals and satisfy almost everywhere on Γ the boundary condition (5).

Consider the sequence of matrices

$$\varphi_m(z) = \frac{1}{2\pi i}\int_\Gamma \frac{a_1(t)\varphi^-_{m-1}(t)}{t-z}\,dt + \frac{1}{2\pi i}\int_\Gamma \frac{b(t)}{t-z}\,dt,$$

where $\varphi_0^-(t) = 0$. It is clear that $\varphi_m^-(t) \in L_p(\Gamma)$, $m = 1, 2, \dots$, if $b(t) \in L_p(\Gamma)$. Further, by a suitable choice of ε it is easily proved that the sequence $\{\varphi_m^-(t)\}$ converges in the norm of the space $L_p(\Gamma)$ to a matrix $\varphi^-(t) \in L_p(\Gamma)$. Hence, in turn, it follows that for every $z \notin \Gamma$ there exists the limit $\lim \varphi_m(z) = \varphi(z)$, and that this limit is represented by the formula

$$\varphi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{a_1(t)\varphi^-(t)}{t-z}\,dt + \frac{1}{2\pi i}\int_\Gamma \frac{b(t)}{t-z}\,dt. \tag{6}$$

The matrix $\varphi(z)$ defined by this formula is representable by an integral of Cauchy type, belongs to the class $L_p(\Gamma)$, and satisfies the boundary condition (5).

Let us now take $b(t) = a(t)r^{-1}(t)$. Then the matrix $\varphi(z)$ belongs to the class $L_p(\Gamma)$, and

$$\varphi^+(t) = a(t)r^{-1}(t)[\varphi^-(t) + I]. \tag{7}$$

Similarly we obtain that there exists a matrix $\psi(z) \in L_p(\Gamma)$ (p is a fixed sufficiently large number), representable by a Cauchy-type integral, and such that

$$\psi^+(t) = a^{-1}(t)r(t)[\psi^-(t) + I]. \tag{8}$$

From (7) and (8) we easily infer that $\det\varphi^+(z)\cdot\det\psi^+(z) = 1$, $\det[\varphi^-(z) + I]\cdot\det[\psi^-(z) + I] = 1$. The determinant $\det r^{-1}(z)$ can have in E^- a finite number of zeros. As is known, one can always choose such a rational matrix $R(z)$, which will have poles at the points where $\det r^{-1}(z)$ has zeros, and such that $\det[\varphi^+R] \neq 0$ in E^+, $\det[r^{-1}(\varphi^- + I)R] \neq 0$ everywhere in E^-, except, possibly, at the infinitely distant point. It is now easily checked that the matrix

$$X(z) = \begin{cases} \varphi^+(z)R(z), & z \in E^+, \\ r^{-1}(z)[\varphi^-(z) + I]R(z), & z \in E^-, \end{cases}$$

---

* The method used was indicated by G. F. Mandzhavidze in reports at the Third All-Union Mathematical Congress in 1956 (^{4}) and at the Third Conference on the Theory of Functions of a Complex Variable in 1957.

is a canonical matrix for $a(t)$ (for this, X must also be put in normal form at infinity), and moreover $X(z),\ X^{-1}(z) \in L_r(\Gamma)$, where r depends on p and n; if p is taken sufficiently large, then r is also a sufficiently large number.

As was already noted above (formula (3)), all solutions of the homogeneous problem (2) will have the form

$$\Phi(z) = X(z)P(z),$$

where $P(z)$ is an arbitrary matrix whose elements are entire functions. Hence it is not difficult to derive the conclusion that all solutions of the homogeneous problem (and, in particular, the canonical matrices) belong to the class $L_p(\Gamma)$ for any $p > 1$.

The orders of the columns of the canonical matrix with opposite sign, $\varkappa_1, \varkappa_2, \dots, \varkappa_n$, will be called the partial indices, and their sum $\varkappa = \varkappa_1 + \dots + \varkappa_n$ —the total index ([2],[3]); it is not difficult to show that

$$\varkappa = \frac{1}{2\pi}[\arg \det a(t)]_\Gamma$$

(the symbol $[\ ]_\Gamma$ denotes the increment of the expression enclosed in brackets under traversal along Γ in the positive direction).

After problem (2) has been solved for one contour, it presents no difficulty to consider the case when Γ is a union of contours.

4. Finally, let us consider a problem analogous to problem (2), in which the boundary condition is replaced by the condition

$$\Phi^+[\alpha(t)] = a(t)\Phi^-(t) + b(t), \tag{9}$$

where $a(t)$ is a nonsingular continuous matrix; the matrix $b(t) \in L_p(\Gamma)$, $p > 1$; $\alpha(t)$ is a function mapping the line Γ onto itself with preservation of the direction of traversal and having a second derivative satisfying the Hölder condition, with $\alpha'(t) \neq 0$.

Let us find a piecewise-holomorphic function $\omega(z)$, having its principal part at infinity z and satisfying the boundary condition

$$\omega^+[\alpha(t)] = \omega^-(t) \quad \text{on } \Gamma.$$

$\omega(z)$ can be taken in the form (7)

$$\omega(z) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_\Gamma \frac{\rho[\beta(t)]}{t-z}\,dt, & z \in E^+, \\ \dfrac{1}{2\pi i} \displaystyle\int_\Gamma \frac{\rho(t)\,dt}{t-z}, & z \in E^-, \end{cases}$$

where ρ is the solution of the Fredholm integral equation

$$\rho(t_0) + \frac{1}{2\pi i} \int_\Gamma \left[\frac{a'(t)}{a(t)-a(t_0)} - \frac{1}{t-t_0} \right] \rho(t)\,dt = t_0;$$

$\beta(t)$ is the function inverse to $\alpha(t)$.

It can be shown that $\omega^+(z)$ maps the domain E^+ one-to-one onto a certain simply connected domain E_1^+; $\omega^-(z)$ maps the domain E^- one-to-one onto an (infinite) domain E_1^-, having no interior points in common with E_1^+; the domains E_1^+ and E_1^- are separated by a common boundary Γ_1, which is a Lyapunov curve.

Consider the boundary-value problem

$$\Phi_1^+(\tau) = a_1(\tau)\Phi_1^-(\tau) + b_1(\tau), \qquad \tau \in \Gamma_1,$$

where $a_1(\tau) = a(\sigma(\tau))$, $b_1(\tau) = b(\sigma(\tau))$; $\sigma(\tau)$ is the function inverse to $\omega^-(t)$. Then the solutions of problem (9) are given by the formulas

$$\Phi^+(z) = \Phi_1^+(\omega^+(z)), \qquad \Phi^-(z) = \Phi_1^-(\omega^-(z)).$$

We have reduced problem (9) to a problem of the form (2); by generalizing the known method for studying problem (9), one can investigate it under weaker restrictions imposed on $\alpha(t)$.

Tbilisi Mathematical Institute
named after A. M. Razmadze
Academy of Sciences of the Georgian SSR

Received
7 VII 1958

References

1. B. V. Khvedelidze, *Proceedings of the Tbilisi Mathematical Institute named after A. M. Razmadze, Academy of Sciences of the Georgian SSR*, **23** (1956).

2. N. I. Muskhelishvili, *Singular Integral Equations*, Moscow-Leningrad, 1946.

3. N. P. Vekua, *Systems of Singular Integral Equations*, Moscow–Leningrad, 1950.
4. *Proceedings of the Third All-Union Mathematical Congress*, **1**, 1956, p. 88.
5. F. D. Gakhov, *Uspekhi Mat. Nauk*, **7** (50), no. 4 (1952).
6. G. Birkhoff, *Proc. Am. Acad. Arts and Sci.*, **49** (1913).
7. D. A. Kveselava, *DAN*, **53**, no. 8 (1946).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.