

**EXPANSION
THEOREMS
CONNECTED WITH
BOUNDARY-VALUE
PROBLEMS FOR THE
EQUATION $(u_{xx} -$
 $K(x)u_{tt} = 0)$ IN THE
STRIP $(0 \leq x \leq 1)$
WITH DEGENERATION
OR SINGULARITY AT
THE BOUNDARY**

Problem (C)

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Abstract

Full Text

MATHEMATICS

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EXPANSION THEOREMS CONNECTED WITH BOUNDARY-VALUE PROBLEMS FOR THE EQUATION $u_{xx} - K(x)u_{tt} = 0$ IN THE STRIP $0 \leq x \leq 1$ WITH DEGENERATION OR SINGULARITY AT THE BOUNDARY

(Presented by Academician V. I. Smirnov on 4 III 1958)

In ⁽¹⁾ the following mixed problem was solved by the Fourier method:

Problem (C)

$$u_{xx} = -K(x)u_{tt} = 0; \tag{1}$$

$$u|_{t=l(x)} = p(x), \quad x \in [0, 1];$$

$$u_t|_{t=l(x)} \begin{cases} = q(x), & \text{if } |l'(x)| < \sqrt{K(x)}, \quad x \in [0, 1], \\ \text{is not prescribed,} & \text{if } l'(x) = \sqrt{K(x)}, \quad x \in [0, 1]; \end{cases}$$

$$u(0, t) \cos \xi + u_x(0, t) \sin \xi = 0, \quad 0 < \xi \leq \pi,$$

$$u(1, t) \cos \eta + u_x(1, t) \sin \eta = 0, \quad 0 < \eta \leq \pi.$$

It was required here that the function $K(x)$ be twice differentiable and positive on the closed interval $[0, 1]$.

We shall consider here problem (C) under the assumption that

$$K(x) = x^\alpha(1-x)^\beta K_0(x), \tag{2}$$

where $K_0(x)$ is a positive twice differentiable function on $[0, 1]$, and

$$\alpha > -1 \quad \text{for } \xi \neq \pi, \quad \beta > -1 \quad \text{for } \eta \neq \pi, \tag{3}$$

$$\alpha > -2 \quad \text{for } \xi = \pi; \quad \beta > -2 \quad \text{for } \eta = \pi.$$

The tending of $K(x)$ to zero or infinity at the endpoints of the interval $[0, 1]$ substantially complicates the problem, since the asymptotic expansions used in ⁽¹⁾, uniformly on the whole interval $[0, 1]$, are not applicable here.

Under the indicated conditions, the singular Sturm-Liouville problem

$$B_n'' + \lambda^2 K B_n = 0; \tag{4}$$

$$B_n(0) \cos \xi + B_n'(0) \sin \xi = 0, \quad B_n(1) \cos \eta + B_n'(1) \sin \eta = 0 \tag{5}$$

has a discrete spectrum, investigated in the work of A. A. Dorodnitsyn ⁽²⁾. Suppose that zero is not an eigenvalue and that the $B_n(x)$ are normalized. We seek the solution of problem (C) in the form

$$u(x, t) = \sum_{n=-\infty}^{+\infty} c_n B_n(x) \exp(-i\lambda_n t), \tag{6}$$

where λ_n, B_n are defined by (4)–(5), $\lambda_{-n} = -\lambda_n$.

Under the conditions on $t = l(x)$,

$$p(x) \approx \sum_{n=-\infty}^{+\infty} c_n B_n(x) \exp\{-i\lambda_n l(x)\}; \tag{7}$$

$$q(x) \approx \sum_{n=-\infty}^{+\infty} (-i\lambda_n) c_n B_n(x) \exp[-i\lambda_n l(x)]; \tag{8}$$

the coefficients c_n in (6), with the aid of the generalized orthogonality relation (1), are formally determined in the form

$$c_n = \frac{i}{2\lambda_n} \int_0^1 e^{i\lambda_n l} \{B_n[p'l' + q(K - l'^2) - Ki\lambda_n p] - B_n' p l'\} dx. \tag{9}$$

In the case $l' \equiv \sqrt{K}$, integration by parts, under the assumption that $p\sqrt{K} \rightarrow 0$ as $x \rightarrow 0$ and $x \rightarrow 1$, leads to the formula

$$c_n = -\frac{i}{\lambda_n} \int_0^1 e^{i\lambda_n l} p \left\{ \sqrt{K} B_n' + B_n \left[\frac{K'}{4\sqrt{K}} + i\lambda_n K \right] \right\} dx. \tag{10}$$

Let us denote the partial sums of the series (7)–(8) with coefficients (9) respectively by $S_n^{(p)}(x)$ and $S_n^{(q)}(x)$, and the partial sum of the series (7) with coefficients (10) by $\Sigma_n^{(p)}(x)$.

Under the condition $\alpha < 2$, $\beta < 2$ (together with (3)), the following expansion theorems are valid.

Theorem 1. Suppose $|l'(x)| < \sqrt{K(x)}$, $x \in [0, 1]$, and the functions l', p', q are representable in the form

$$l'(x) = x^{\alpha/2}(1-x)^{\beta/2}l_1(x), \quad p'(x) = x^{a-1}(1-x)^{b-1}p_1(x),$$

$$q(x) = x^{a-1-\alpha/2}(1-x)^{b-1-\beta/2}q_0(x),$$

where l_1, p_1, q_0 are bounded on $[0, 1]$.* Suppose, furthermore,

$$p(0) = 0 \quad \text{for } \alpha \leq 0 \quad \text{in the case } \xi = \pi,$$

$$p(1) = 0 \quad \text{for } \beta \leq 0 \quad \text{in the case } \eta = \pi.$$

Then, if

$$a > -\frac{\alpha}{4} \quad \text{for } \alpha \geq 0, \quad b > -\frac{\beta}{4} \quad \text{for } \beta \geq 0,$$

$$a > -\frac{\alpha}{2} \quad \text{for } \alpha \leq 0; \quad b > -\frac{\beta}{2} \quad \text{for } \beta \leq 0,$$

then at every point $x \in (0, 1)$, in a neighborhood of which the functions

$$Q_{\pm} = q(\sqrt{K} \mp l') \pm p'$$

have bounded variation,

$$S_n^{(p)}(x) \xrightarrow{n \rightarrow \infty} p(x), \quad S_n^{(q)}(x) \xrightarrow{n \rightarrow \infty} \frac{q(x-0) + q(x+0)}{2}.$$

Theorem 2. Suppose

$$|l'(x)| < \sqrt{K(x)}, \quad x \in (0, 1); \quad l'(x) = x^{\alpha/2}(1-x)^{\beta/2}l_1(x),$$

$$p(x) = x^a(1-x)^b p_0(x);$$

$l_1(x), p_0(x)$ are bounded on $[0, 1]$.

Then, if

$$a > -1 - \frac{\alpha}{4} \quad \text{for } \alpha \geq 0, \quad b > -1 - \frac{\beta}{4} \quad \text{for } \beta \geq 0,$$

$$a > -1 - \frac{\alpha}{2} \quad \text{for } \alpha \leq 0; \quad b > -1 - \frac{\beta}{2} \quad \text{for } \beta \leq 0,$$

* Here and below all functions are assumed measurable.

then at each point $x \in (0, 1)$, in a neighborhood of which $p(x)$ has bounded variation,

$$\sum_{n=-\infty}^{\infty} c_n B_n(x) \exp[-i\lambda_n l(x)] = \frac{p(x-0) + p(x+0)}{2},$$

where

$$c_n = -\frac{i}{2\lambda_n} \int_0^1 e^{i\lambda_n l} p(l' B_n' + i\lambda_n K B_n) dx.$$

Theorem 3. Suppose

$$l(x) = \zeta(x) = \int_0^x \sqrt{K(x)} dx, \quad p(x) = x^a (1-x)^b p_0(x),$$

$p_0(x)$ is bounded on $[0, 1]$. Then, if

$$a > -\frac{\alpha}{4} \quad \text{for } \alpha \geq 0, \quad b > -\frac{\beta}{4} \quad \text{for } \beta \geq 0,$$

$$a > -\frac{\alpha}{2} \quad \text{for } \alpha \leq 0; \quad b > -\frac{\beta}{2} \quad \text{for } \beta \leq 0,$$

then at each point $x \in (0, 1)$, in a neighborhood of which $p(x)$ has bounded variation,

$$\Sigma_n^{(p)}(x) \xrightarrow{n \rightarrow \infty} \frac{p(x-0) + p(x+0)}{2}.$$

Theorem 4. Suppose $l(x) \equiv \zeta(x)$, $\alpha \leq 0$, $\beta \leq 0$, $p(x)$ is bounded for $0 < x_1 \leq x \leq x_2 < 1$, and in neighborhoods of the endpoints of the interval $[0, 1]$ there exists $p'(x)$, representable in the form

$$p'(x) = x^{a-1}p_0(x), \quad 0 < x \leq x_1; \quad p'(x) = (1-x)^{b-1}p_1(x), \quad x_2 \leq x < 1,$$

where $p_0(x)$, $p_1(x)$ are bounded respectively on $[0, x_1]$ and $[x_2, 1]$. Suppose further that $p(0) = 0$ for $\xi = \pi$, $p(1) = 0$ for $\eta = \pi$. Then, if $a > -\frac{\alpha}{2}$, $b > -\frac{\beta}{2}$, then at each point $x \in (0, 1)$, in a neighborhood of which $p(x)$ has bounded variation,

$$\sum_{n=-\infty}^{\infty} c_n B_n(x) \exp[-i\lambda_n \zeta(x)] = \frac{p(x-0) + p(x+0)}{2},$$

where

$$\begin{aligned} c_n = & -\frac{i}{\lambda_n} \int_{x_1}^{x_2} e^{i\lambda_n \zeta} p \left\{ B'_n \sqrt{K} + B_n \left[\frac{K'}{4\sqrt{K}} + i\lambda_n K \right] \right\} dx + \\ & + \frac{1}{2\lambda_n^2} \left\{ p(1-) \cos \eta e^{i\lambda_n \zeta(1)} - p(0+) \cos \xi + \left[e^{i\lambda_n \zeta} p(B'_n + i\lambda_n \sqrt{K} B_n) \right]_{x_1}^{x_2} + \right. \\ & \left. + \int_0^{x_1} e^{i\lambda_n \zeta} p' [B'_n + i\lambda_n \sqrt{K} B_n] dx + \int_{x_2}^1 e^{i\lambda_n \zeta} p' [B'_n + i\lambda_n \sqrt{K} B_n] dx \right\}. \end{aligned}$$

The proof is carried out by the method of contour integration over an infinitely large circle in the complex λ -plane, using asymptotic formulas for the solutions of equation (4), obtained on the basis of

work of A. A. Dorodnitsyn¹. In an interval containing one of the endpoints of $[0, 1]$ (say, zero), the solutions of equation (4) are asymptotically representable in terms of generalized Airy functions of the argument $s = \lambda^{\frac{2}{\alpha+2}} \omega(x)$, where

$$\omega(x) = \left\{ \frac{\alpha+2}{2} \int_0^x \sqrt{K(x)} dx \right\}^{\frac{2}{\alpha+2}}, \quad \omega'(0) \neq 0.$$

An essential idea in the proof is the selection of a shrinking neighborhood of the endpoint $[0, \gamma]$, where γ is defined by the equality

$$\left| \lambda^{\frac{2}{\alpha+2}} \omega(\gamma) \right| = N = \text{const.}$$

¹A. A. Dorodnitsyn, *Uspekhi Mat. Nauk*, **7**, No. 6 (1952).

Outside $[0, \gamma]$, the generalized Airy functions admit an asymptotic representation in terms of \sin , \cos .

After the expansions (7)–(8) have been justified, the investigation of the series (6) from the point of view of satisfying, in some sense, the remaining conditions of problem (C) is carried out with the aid of the asymptotic expansion of the coefficients c_n .

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REFERENCES

Note: Figure translations are in progress. See original paper for figures.

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