PLANE DEFORMATION OF AN INCOMPRESSIBLE MATERIAL

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Abstract

Full Text

THEORY OF ELASTICITY

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PLANE DEFORMATION OF AN INCOMPRESSIBLE MATERIAL

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1. The position of the points of a body in the natural state is determined by Cartesian coordinates x_1, x_2, x_3. Let the displacement vector of a particle have the components $u_1(x_1, x_2)$, $u_2(x_1, x_2)$, $u_3 = 0$. Then the deformed state of the neighborhood of a point can be determined ([1]) by the components of the tensor $((\varepsilon_{ik}))$ and the vector $\vec{\omega}$, of which only ε_{11}, ε_{22}, ε_{12}, and ω_3 may differ from zero.

If ϑ is the angle between the first of the principal fibers in the natural state and the vector $\mathbf{i}_1$; λ_1, λ_2 are the elongations of the principal fibers; ω is the angle of rotation of the neighborhood of the point as a rigid body, then

$$\begin{aligned} 2(1+\varepsilon_{11}) &= (\lambda_1+\lambda_2)\cos\omega + (\lambda_1-\lambda_2)\cos(2\vartheta+\omega), \\ 2(1+\varepsilon_{22}) &= (\lambda_1+\lambda_2)\cos\omega - (\lambda_1-\lambda_2)\cos(2\vartheta+\omega), \\ 2(\varepsilon_{12}-\omega_3) &= -(\lambda_1+\lambda_2)\sin\omega + (\lambda_1-\lambda_2)\sin(2\vartheta+\omega), \\ 2(\varepsilon_{12}+\omega_3) &= (\lambda_1+\lambda_2)\sin\omega + (\lambda_1-\lambda_2)\sin(2\vartheta+\omega). \end{aligned} \tag{1}$$

Hence, among other things, it follows that

$$2\omega_3 = (\lambda_1+\lambda_2)\sin\omega.$$

Considering the deformation of an incompressible material, we put

$$\lambda_1 = 1+\eta, \qquad \lambda_2 = (1+\eta)^{-1}, \qquad \lambda_3 = 1,$$

$$\ln\lambda_k = \sqrt{2\mathcal{E}_i}\cos\beta_k, \qquad \beta_1 = \beta, \qquad \beta_2 = \beta + \frac{2}{3}\pi, \qquad \beta_3 = \beta - \frac{2}{3}\pi,$$

whence follow the expressions for the phase and intensity of deformation:

$$\beta = \frac{1}{6}\pi, \qquad \sqrt{1.5\,\mathcal{E}_i} = \ln(1+\eta) = \frac{1}{2}\vartheta. \tag{2}$$

The displacement compatibility conditions reduce to the system:

$$\begin{aligned} 2(\vartheta+\omega)_{,1} &= 2\operatorname{ch}\vartheta\,\vartheta_{,1} - (\operatorname{ch}\vartheta)_{,2} + (\sin 2\vartheta\ \operatorname{sh}\vartheta)_{,1} - (\cos 2\vartheta\ \operatorname{sh}\vartheta)_{,2},\\ 2(\vartheta+\omega)_{,2} &= 2\operatorname{ch}\vartheta\,\vartheta_{,2} + (\operatorname{ch}\vartheta)_{,1} - (\sin 2\vartheta\ \operatorname{sh}\vartheta)_{,2} - (\cos 2\vartheta\ \operatorname{sh}\vartheta)_{,1}, \end{aligned} \tag{3}$$

and the compatibility condition for rotations of the elements of the body proves to be the equation

$$(\cos 2\vartheta\ \operatorname{sh}\vartheta)_{,22} - (\cos 2\vartheta\ \operatorname{sh}\vartheta)_{,11} - 2(\sin 2\vartheta\ \operatorname{sh}\vartheta)_{,12} + \nabla^2 \operatorname{ch}\vartheta + 2\vartheta_{,2}(\operatorname{ch}\vartheta)_{,1} - 2\vartheta_{,1}(\operatorname{ch}\vartheta)_{,2} = 0, \tag{4}$$

where ∇^2 denotes the Laplace operator.

2. The nonlinear equilibrium equations are written in generalized stresses ([2]), which are expressed in terms of the hydrostatic σ and octahedral tangential τ stresses. Assuming the body to be isotropic, the principal directions of stresses and deformations to coincide, the phases of the true stresses and logarithmic elongations to be the same, and the intensity of deformations, known from experiments, to be a single-valued function only of the octa-

of the reduced shear stress, we satisfy the equations of equilibrium in the absence of body forces by introducing the stress function U:

$$\frac{\sigma}{\sqrt{1,5}} + \tau\cos 2\vartheta - f = pU_{,22}, \qquad \frac{\sigma}{\sqrt{1,5}} - \tau\cos 2\vartheta - f = pU_{,11},$$
$$\tau\sin 2\vartheta = -pU_{,12}, \qquad f = 2\sqrt{1,5}\int_0^{\vartheta_i} \tau(\vartheta_i)\, d\vartheta_i, \tag{5}$$

where p is a characteristic stress.

Putting $V = \tau^{-1}\operatorname{sh}\vartheta$, on the basis of (5) and (4) we obtain the resolving equation of the problem

$$\begin{aligned} &\nabla^4 U + \frac{1}{p}\frac{d\vartheta}{d\tau}\nabla^2 t^2 + \frac{1}{2\tau V}\frac{dV}{d\tau}\left[(\tau^2_{,11} - \tau^2_{,22})(U_{,11} - U_{,22}) + 4\tau^2_{,12}U_{,12}\right]\\ &\quad + \frac{1}{2\tau V}\frac{d}{d\tau}\left(\frac{V}{p}\frac{d\vartheta}{d\tau}\right)\left[(\tau^2_{,1})^2 + (\tau^2_{,2})^2\right] + p\frac{d\vartheta}{d\tau}\left[U_{,122}(U_{,111} - U_{,122}) + \right.\\ &\quad\quad \left. + U_{,112}(U_{,222} - U_{,211})\right] + \frac{1}{\tau V}\frac{dV}{d\tau}\left(\tau^2_{,1}\nabla^2 U_{,1} + \tau^2_{,2}\nabla^2 U_{,2}\right)\\ &\quad + \frac{1}{4\tau V}\frac{d}{d\tau}\left(\frac{1}{\tau}\frac{dV}{d\tau}\right)\left\{\left[(\tau^2_{,1})^2 - (\tau^2_{,2})^2\right](U_{,11} - U_{,22}) + 4\tau^2_{,1}\tau^2_{,2}U_{,12}\right\} = 0. \end{aligned} \tag{6}$$

On passing to the relations of the classical theory of elasticity, this equation reduces to the biharmonic equation.

3. Let on a boundary element with normal $\vec{\nu}(\cos\alpha, \sin\alpha)$ in the natural state the true normal $p\sigma_n$ and tangential $p\tau_n$ stresses be prescribed. Since

$$p\lambda_\sigma^2\sigma_n = \lambda_\sigma^2\sigma - \sqrt{1,5}\tau[\operatorname{sh}\vartheta - \operatorname{ch}\vartheta\cos 2(\alpha-\vartheta)],$$

$$p\lambda_\tau^2\tau_n = -\sqrt{1,5}\tau\sin 2(\alpha-\vartheta), \qquad \lambda_\tau^2 = \operatorname{ch}\vartheta - \operatorname{sh}\vartheta\cos 2(\alpha-\vartheta),$$

on the basis of (5) we find the boundary conditions

$$\frac{dU_{,1}}{ds} = A\cos\alpha - B\sin\alpha, \qquad \frac{dU_{,2}}{ds} = A\sin\alpha + B\cos\alpha, \tag{7}$$

where s is the length of the arc of the boundary contour Γ in the natural state,

$$A = -\frac{1}{\sqrt{1,5}}\lambda_\tau^2\big|_\Gamma\tau_n,$$

$$B = -\frac{1}{p}f\big|_\Gamma + \frac{1}{\operatorname{ch}\vartheta}\left\{\frac{\lambda_\tau^2\sigma_n}{\sqrt{1,5}} + \frac{V\tau^2}{p} + pV\left(\frac{1}{2}\nabla^2 U + \frac{f}{p}\right)\left[\frac{1}{2}\cos 2\alpha\left(U_{,22} - U_{,11}\right) - \sin 2\alpha\, U_{,12}\right]\right\}\Bigg|_\Gamma. \tag{8}$$

4. In investigating finite deformations of nonferrous metals one may take

$$\tau = G\ \operatorname{th}\vartheta, \qquad G = \text{const}. \tag{9}$$

We denote

$$\mu = \frac{p}{G}, \qquad t = (U_{,12})^2 - \frac{1}{4}(U_{,22} - U_{,11})^2, \qquad H = \frac{1}{\operatorname{ch}^2\vartheta} = 1 - \mu^2 t \tag{10}$$

and write the resolving equation in the form

$$\nabla^2 U + \sum_{k=1}^{4}\mu^k L_k(U) = 0, \tag{11}$$

where L_k are nonlinear operators of degree $k+1$, for example:

$$L_1(U) = \nabla^2 t + U_{,122}\left(U_{,111} - U_{,122}\right) + U_{,112}\left(U_{,222} - U_{,112}\right). \tag{12}$$

Since in this case

$$f = -G \ln H, \qquad \frac{\sigma}{\sqrt{1,5}} = f + \frac{1}{2} p \nabla^2 U,$$

it is easy to find:

$$A = -\frac{\tau_n}{\sqrt{1,5H}} \left(1 - \mu K \sqrt{H}\right), \qquad K = \frac{1}{2} \cos 2\alpha \left(U_{,22} - U_{,11}\right) - \sin 2\alpha \, U_{,12},$$

$$B = \frac{\sigma_n}{\sqrt{1,5}} + \mu \left(\frac{1}{\mu^2} \ln H + t\sqrt{H}\right) - \mu K \sqrt{H} \left(\frac{\sigma_n}{\sqrt{1,5}} - \frac{1}{2} \nabla^2 U + \frac{1}{\mu} \ln H\right). \quad (13)$$

The structure of equation (11) and of the boundary conditions (7), (13) indicates the possibility of representing the solution of the problem by an expansion in powers of the small parameter μ:

$$U = U^{(0)} + \mu U^{(1)} + \mu^2 U^{(2)} + \ldots \quad (14)$$

The determination of $U^{(0)}$ is equivalent to the solution of the classical problem ([3]), while the computation of each of the subsequent approximations reduces to the solution of problems of the type

$$\nabla^4 U^{(1)} = -L_1 \left(U^0\right),$$

$$\left.\frac{dU^{(1)}_{,1}}{ds}\right|_{\Gamma} = f_1 \left(U^0\right), \qquad \frac{dU^{(1)}_{,2}}{ds} = f_2 \left(U^0\right), \quad (15)$$

therefore, for the solution of concrete problems one may successfully apply the methods developed by N. I. Muskhelishvili.

In contrast to the known ([4]) methods of formulating and solving plane problems of the nonlinear theory of elasticity, here the magnitudes of deformations and displacements are not restricted; therefore, the computation of any approximation and the analysis of the convergence of (14) are meaningful.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.