Corresponding Member of the Academy of Sciences of the USSR Ya. B. ZELDOVICH and G. I. BARENBLATT

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Abstract

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Corresponding Member of the Academy of Sciences of the USSR Ya. B. ZELDOVICH and G. I. BARENBLATT

ON THE ASYMPTOTIC PROPERTIES OF SELF-SIMILAR SOLUTIONS OF THE EQUATIONS OF UNSTEADY GAS FILTRATION

1°. In numerous nonlinear problems of mathematical physics, obtaining and analyzing exact self-similar solutions constitutes a very effective method of investigation. It is important to bear in mind that self-similar solutions are of interest not only, and not so much, as particular solutions of separate narrow classes of problems, but chiefly as an instrument for obtaining approximate solutions and asymptotic representations of solutions of more general classes of problems. In particular, in a number of cases self-similar solutions determine the asymptotic behavior, for large values of time, of solutions of sufficiently broad classes of unsteady problems. Profound considerations on this subject are contained in the book of L. I. Sedov (^{1}). As is known, the existence of self-similar solutions is connected with the invariance of the basic equations of the problem with respect to some group of transformations. It turns out to be possible, by using this invariance, to indicate (also in self-similar form) the following terms of the asymptotic representations of the solution for large values of time. In the present note such results are obtained as applied to the Cauchy problem for the nonlinear equation of unsteady gas filtration (coinciding with the heat-conduction equation for a power-law dependence of the thermal conductivity on temperature), whose self-similar solutions corresponding to an initial condition in the form of a δ-function were obtained earlier (^{2}, ^{3}). Despite a certain formal difference, the considerations set forth in the present work are closely connected with the work (^{4}), where the equation of heat propagation of a flame, invariant with respect to a narrower group of continuous transformations, was considered. We note that, in the spherical case, the problem of perturbations of a self-similar solution was considered in the work of E. I. Andriankina and O. S. Ryzhov (^{5}).

2°. Let us consider the asymptotic behavior of solutions of the Cauchy problem for L. S. Leibenzon' s equation of unsteady gas filtration

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u^{n+1}}{\partial x^2} \quad \text{or} \quad \frac{\partial w^{1/(n+1)}}{\partial t} = a^2 \frac{\partial^2 w}{\partial x^2}; \qquad w = u^{n+1} \tag{1}$$

(u is the gas density, a^2 is a constant depending on the properties of the medium and of the gas, x is the coordinate ($-\infty < x < \infty$), t is time, n is the polytropic exponent), corresponding to bounded initial distributions $u(x,0) = U(x)$ that vanish outside some finite interval of the x-axis.

For illustration, let us first consider the linear case ($n = 0$). As is known, the solution of the Cauchy problem under consideration in this case is represented in the form

$$u(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} U(y) e^{-(x-y)^2/4a^2 t}\, dy. \tag{2}$$

It is not hard to show, by expanding the kernel of the integrand, that for large t this solution is represented in the form

$$u(x,t) = -\frac{1}{2a\sqrt{\pi t}} e^{-\xi^2} \int_{-\infty}^{\infty} U(y)\, dy + \frac{\xi e^{-\xi^2}}{2a^2\sqrt{\pi t}} \int_{-\infty}^{\infty} U(y) y\, dy +$$
$$+ \frac{e^{-\xi^2}(2\xi^2 - 1)}{8a^3\sqrt{\pi}\, t^{3/2}} \int_{-\infty}^{\infty} U(y) y^2 dy + \ldots \qquad \left(\xi = \frac{x}{2a\sqrt{t}}\right), \tag{3}$$

i.e., in the form of a sum of self-similar terms in which the powers of time each time increase in absolute value by 1/2, while the coefficients are expressed through successive moments of the initial distribution. Expression (3) confirms the intuitively obvious fact that the general solution (2) tends to the self-similar solution

$$u_0'(x,t) = \frac{E}{2a\sqrt{\pi t}}\, e^{-x^2/4a^2 t}$$

(provided that the first moment of the function $U(x)$ does not accidentally vanish). In the linear case it is also easy to verify that, with a suitable choice of the normalizing factor E, the coordinate shift x_0, and the time shift ϑ, the self-similar function

$$u_0(x - x_0, t + \vartheta, E) = \frac{E}{2a\sqrt{\pi(t+\vartheta)}} e^{-(x-x_0)^2/4a^2(t+\vartheta)}$$

differs from the exact solution by terms of order not lower than $u_0'/t^{3/2}$. For this one must take

$$E = \int_{-\infty}^{\infty} U(y)\, dy, \qquad x_0 = \int_{-\infty}^{\infty} U(y) y\, dy \Big/ \int_{-\infty}^{\infty} U(y)\, dy,$$

$$\vartheta = \int_{-\infty}^{\infty} U(y) y^2 dy \Big/ a^2 \int_{-\infty}^{\infty} U(y)\, dy.$$

3°. Let us now turn to the nonlinear case ($n \neq 0$). First of all, note that the solution of the problem under consideration satisfies the relations

$$\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} U(y)\,dy = E, \qquad \int_{-\infty}^{\infty} u(x,t)x\,dx = \int_{-\infty}^{\infty} U(y)y\,dy = x_0 E, \tag{4}$$

where x_0 and E are certain constants. The self-similar solution of equation (1), satisfying conditions (4), has the form ([2,3])

$$w = w_0 = \frac{E^{2(n+1)/(n+2)}}{[a^2(t+\vartheta)]^{(n+1)/(n+2)}} \left[\frac{n}{2(n+1)(n+2)}\right]^{(n+1)/n} (\xi_0^2 - \xi^2)^{(n+1)/n} \qquad (|\xi| \leq \xi_0),$$

$$w_0 \equiv 0 \qquad (|\xi| \geq \xi_0), \tag{5}$$

where

$$\xi = \frac{x - x_0}{[E^n a^2(t+\vartheta)]^{1/(n+2)}}; \quad \xi_0 = \xi_0(n)$$

is a certain nonzero function of n; for small n, $\xi_0 \sim n^{-1/2}$; ϑ is an arbitrary constant. For large t the solution of the general problem under consideration is represented in the form

$$w(x,t) = w_0(x - x_0, t+\vartheta, E) + \varphi(\xi,\tau), \qquad \tau = \ln(t+\vartheta), \tag{6}$$

where $\varphi(\xi,\tau)$ is regarded as small in comparison with w_0; the choice of the constant ϑ will be discussed below. The problem consists in determining the order of decrease of the function $\varphi(\xi,\tau)$ with time. Substituting expression (6) into equation (1) and linearizing, we obtain for φ the linear equation

$$\frac{\partial\varphi}{\partial\tau} = L_\xi\varphi = \frac{n}{2(n+2)}(1-\xi^2)\frac{\partial^2\varphi}{\partial\xi^2} + \frac{1}{n+2}\xi\frac{\partial\varphi}{\partial\xi} + \frac{2}{(n+2)(1-\xi^2)}\varphi - \varphi, \tag{7}$$

where $\zeta = \xi/\xi_0$. We seek the solution of this equation by the method of separation of variables in the form

$$\varphi = \sum^{(n)} e^{-\lambda_n\tau} f(\zeta,\lambda_n) = \sum^{(n)} (t + \vartheta^{-\lambda_n}) f(\zeta,\lambda_n),$$

where λ_n are the eigenvalues of the operator L_ζ. To determine several of the first eigenvalues we shall make use of the invariance indicated above, i.e., of the fact that the function $w = w_0(x - x_0 + \Delta,\ t + \vartheta + \sigma,\ E + \varepsilon)$ is also a solution of equation (1) for arbitrary $\Delta, \sigma, \varepsilon$. Up to small quantities of second order in $\Delta, \sigma, \varepsilon$, this solution is represented in the form

$$w_0(x - x_0,\ t + \vartheta,\ E) + \varepsilon\,\frac{\partial w_0}{\partial E} + \Delta\,\frac{\partial w_0}{\partial x} + \sigma\,\frac{\partial w_0}{\partial t}. \tag{8}$$

It is easy to see that, owing to the self-similarity of $w_0(x, t, E)$, the last three terms of expression (8) are represented in the form $(t + \vartheta)^{-m} f(\zeta)$, satisfy equation (7), and, consequently, the corresponding functions $f(\zeta)$ are eigenfunctions of the operator L_ζ, corresponding, as is not hard to verify, respectively to

$$\lambda_0 = \frac{n+1}{n+2}, \qquad \lambda_1 = 1, \qquad \lambda_2 = \frac{2n+3}{n+2}.$$

But it is easy to show that $f\bigl(\zeta, \frac{n+1}{n+2}\bigr)$, corresponding to the everywhere positive $\partial w_0/\partial E$, does not vanish anywhere inside the interval $(-1 < \zeta < 1)$; $f(\zeta, 1)$, corresponding to the function $\partial w_0/\partial x$, which changes sign once, vanishes inside this interval once; $f\bigl(\zeta, \frac{2n+3}{n+2}\bigr)$, corresponding to the function $\partial w_0/\partial t$, which changes sign twice, vanishes inside the interval twice. Therefore these eigenfunctions are, respectively, the first, second, and third eigenfunctions of the operator L_ζ, and, consequently, the asymptotic representation (6) is written in the form

$$w(x,t) = w_0(x - x_0,\ t + \vartheta,\ E) + \frac{c_0}{(t+\vartheta)^{(n+1)/(n+2)}}\, f\Bigl(\zeta, \frac{n+1}{n+2}\Bigr) +$$

$$+ \frac{c_1}{(t+\vartheta)}\, f(\zeta, 1) + \frac{c_2}{(t+\vartheta)^{(2n+3)/(n+2)}}\, f\Bigl(\zeta, \frac{2n+3}{n+2}\Bigr) + \cdots. \tag{9}$$

The conditions (4) give that the coefficients c_0 and c_1 are equal to zero, so that

$$w(x,t) = w_0(x - x_0,\ t + \vartheta,\ E) +$$

$$+ \frac{c_2}{(t+\vartheta)^{(2n+3)/(n+2)}}\, f\Bigl(\zeta, \frac{2n+3}{n+2}\Bigr) + o\bigl(t^{-(2n+3)/(n+2)}\bigr). \tag{10}$$

It is obvious that there exists such a ϑ for which the coefficient c_2 also vanishes, so that, with such a choice of ϑ, the solution $w(x,t)$ will differ from the function $w_0(x - x_0, t + \vartheta, E)$ by a quantity

$$o\bigl(t^{-(2n+3)/(n+2)}\bigr).$$

4°. It should be noted that in the solution of the nonlinear problem there is a boundary of the perturbed region, which gives a singularity of the solution. In the presence of a perturbation, the boundary is also shifted by an amount proportional to the perturbation; it is not difficult to find this by using the well-known method of Lighthill (^{6}). We also note that in the problem under consideration it is possible to find the entire spectrum of the operator L_ζ, using the fact that the eigenfunctions of this operator are expressed in terms of Legendre functions of order $\frac{1}{n} - 1$ (it is quite curious to trace how, as $n \to 0$, these functions go over into Hermite polynomials, corresponding to the linear case). We have

$$\lambda_m = \frac{m+1}{n+2} + \frac{[m(m-1)+2]\,n}{2(n+2)}, \qquad m = 0, 1, \dots .$$

The structure of the spectrum found confirms I. M. Gelfand' s hypothesis that, for a broad class of self-similar problems, the eigenvalues are represented as integer combinations of several numbers (in the present case, of the two numbers $1/(n+2)$ and $n/(n+2)$).

In a completely analogous way one determines the asymptotic character of self-similar solutions corresponding to various mixed boundary-value problems, in particular, solutions of the dipole type corresponding to the zero boundary condition on the boundary of a semi-infinite reservoir [7].

In the spherical case, as shown in [5], perturbations can be classified according to their angular dependence $P_{lm}(\theta, \varphi)$ (spherically symmetric $l = 0$, dipole $l = 1$, quadrupole $l = 2, \ldots$). Variation of the energy gives the lowest value $\lambda = 0$ for $l = 0$, a shift of the coordinate gives λ' with $l = 1$, and a shift of time gives the following λ'' with $l = 0$. In this case $|\lambda''| > |\lambda'|$. Concerning λ''' with $l = 2$, one can assert only that $|\lambda'''| > |\lambda'|$, but the relation between λ''' and λ'' is unknown.

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Oil Institute
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.