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Abstract

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GROUPS AND INVARIANT-GROUP SOLUTIONS OF DIFFERENTIAL EQUATIONS

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The important role played by the group properties of differential equations in the study of the structure of the set of their solutions is well known. These properties prove useful in such questions as finding particular solutions with prescribed features, transforming equations for the purpose of separating variables, finding functionally invariant solutions, and so on.

In the present work attention is drawn to the possibility of computing the group for a given system of partial differential equations; the concept of invariant-group solutions is also introduced, and a general method for finding them is indicated. The discussion is carried out within the framework of the classical Lie theory, so that all groups and differential equations are assumed to be analytic, and the properties local. We use the usual tensor notation of summation over repeated indices, as well as the convention of denoting by a letter without indices the collection of quantities denoted by the same letter with different indices.

1. The principal group and its computation. Let a system of differential equations (S) be given for $m \geqslant 1$ unknown functions u^k $(k = 1, 2, \dots, m)$ of $n - m \geqslant 1$ independent variables x^i $(i = 1, 2, \dots, n-m)$. The collection of quantities (x, u) is regarded as a set of coordinates of a point of the n-dimensional space E_n.

Definition 1. A local Lie group G of transformations of E_n is called the **principal group** of the system (S) if: 1) the transformations G take any solution of (S) again into some solution of (S), and 2) every one-parameter local group of transformations of E_n whose transformations have property 1) belongs to G.

Since any system (S), by introducing additional unknown functions and differentiating, can be replaced by an equivalent system of quasilinear first-order partial differential equations, it is sufficient to learn how to compute the principal group for systems of this kind. Let (S) have this form, namely

$$F^\alpha(x, u, p) \equiv f_k^{\alpha i}(x, u) p_i^k + g^\alpha(x, u) = 0 \quad (\alpha = 1, 2, \dots, A), \tag{S}$$

where the notation for the derivatives $p_i^k = \partial u^k / \partial x^i$ has been introduced.

The group G is completely determined by its Lie algebra of infinitesimal operators

$$X = \xi_x^i \frac{\partial}{\partial x^i} + \xi_u^k \frac{\partial}{\partial u^k}$$

or by the isomorphic Lie algebra of prolonged operators

$$\widetilde{X} = \xi_x^i \frac{\partial}{\partial x^i} + \xi_u^k \frac{\partial}{\partial u^k} + \xi_{p_i^k}^k \frac{\partial}{\partial p_i^k}.$$

The quantities ξ_x, ξ_u are functions of (x, u), while ξ_p are functions of (x, u, p); it is convenient to call them the coordinates of these operators.

In terms of the auxiliary operators

$$D_i = \frac{\partial}{\partial x^i} + p_i^k \frac{\partial}{\partial u^k} \qquad (i = 1, 2, \dots, n - m)$$

the coordinates ξ_p of the prolonged operator $\widetilde{X}$ are expressed in terms of the coordinates ξ_x, ξ_u of the operator X by the formulas

$$\xi_{pi}^k = D_i \xi_u^k - p_j^k D_i \xi_x^j$$

$$(k = 1, 2, \dots, m;\ i = 1, 2, \dots, n - m).$$

The group G will be the principal group of the system (S) if and only if the functions F are relative invariants of the first prolongation $\widetilde{G}$ of the group G, i.e. if $\widetilde{X}F = 0$ for every system of values (x, u, p) satisfying (S). Written out in detail, these invariance conditions have the form:

$$f_k^{\alpha i} \frac{\partial \xi_u^k}{\partial x^i} + \xi_x^i \frac{\partial g^\alpha}{\partial x^i} + \xi_u^k \frac{\partial g^\alpha}{\partial u^k} +$$

$$+ \left(\xi_x^j \frac{\partial f_k^{\alpha i}}{\partial x^j} + \xi_u^l \frac{\partial f_k^{\alpha i}}{\partial u^l} + f_l^{\alpha i} \frac{\partial \xi_u^l}{\partial u^k} - f_k^{\alpha j} \frac{\partial \xi_x^i}{\partial x^j} \right) p_i^k$$

$$- f_k^{\alpha i} \frac{\partial \xi_x^j}{\partial u^l} p_j^k p_i^l = 0 \qquad (\alpha = 1, 2, \dots, A). \tag{I}$$

Because the group G acts in E_n, so that the coordinates ξ_x, ξ_u do not depend on the derivatives p, the invariance conditions (I) split and generate a larger number of equations. To construct these new equations we use the following general assumptions concerning the system (S): a) the rank of the matrix $\|f_k^{\alpha i}\|$ (the α are the row numbers) at a point of general position is equal to A (the number of equations (S)), so that A of the variables p, denoted by q_0^α $(\alpha = 1, 2, \dots, A)$, are expressed from the system (S) in terms of the remaining $B = m(n-m) - A$ variables p, denoted by q^β $(\beta = 1, 2, \dots, B)$; b) the variables q can be chosen so that (S) has solutions passing through any point of some domain in the space (x, u, q). These assumptions are satisfied, for example, if (S) can be reduced to Cauchy normal form.

After substituting the quantities q_0, expressed in terms of q from the equations (S), into the conditions (I), the left-hand sides of the latter become nonhomogeneous quadratic forms in the B variables q and must vanish for all values of these variables. In view of this, the conditions (I) split into $\frac{1}{2}(B+1)(B+2)A$ equations, which, according to Lie' s terminology, are called the **determining equations** of the group G. The determining equations constitute a system of linear homogeneous first-order partial differential equations with respect to the unknown coordinates ξ_x, ξ_u of the operator X. The totality of the operators X corresponding to all possible solutions of this system forms the Lie algebra of the principal group G.

The restriction imposed in Definition 1 on the principal group by the requirement that its transformations not lead outside E_n is of fundamental importance, making it possible to reveal the following, previously unnoted fact: in many practically interesting cases one can find the general solution of the determining equations and, thereby, the principal group of the given system (S).

For example, for the system of equations of adiabatic one-dimensional motion of a polytropic gas

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0,$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0, \qquad (S_1)$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \gamma p\frac{\partial u}{\partial x} = 0,$$

where γ is a constant, and u, ρ, p are the desired functions of the independent variables x, t; the Lie algebra of the principal group in the case $\gamma \neq 3$ is generated by the following independent operators:

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial t}, \qquad X_3 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}, \qquad X_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u},$$

$$X_5 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} + 2\rho\frac{\partial}{\partial \rho}, \qquad X_6 = \rho\frac{\partial}{\partial \rho} + p\frac{\partial}{\partial p}.$$

It is interesting to note that for $\gamma = 3$ the principal group of the system (S_1) is broader: its Lie algebra contains, in addition to those listed, one more independent operator

$$X_7 = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} + (x - ut)\frac{\partial}{\partial u} - t\rho\frac{\partial}{\partial \rho} - 3tp\frac{\partial}{\partial p}.$$

Thus the group nature of the well-known fact of the exceptional value $\gamma = 3$ is clarified.

2. Invariant-group solutions

Knowledge of the group of a system (S) makes it possible to find certain collections of particular solutions of (S). The following important concept serves this purpose.

Definition 2. A solution $u = \varphi(x)$ of the system (S) is called an **invariant-group solution** (or an H-solution) if the manifold in E_n defined by the equations $u = \varphi(x)$ is an invariant manifold of some subgroup H of the principal group of this system.

The distinctive feature of H-solutions is that they are determined from a system of equations which, generally speaking, is of a simpler form than the original system (S). This new system, denoted below by the symbol (S/H) (read: S over H), may contain a smaller number of independent variables or unknown functions, may consist of a smaller number of equations, and so on. If the group H is given, then the problem of finding H-solutions reduces to the problem of constructing the corresponding system (S/H).

Below we consider the case when H is intransitive and the invariant manifolds of H coincide with its systems of intransitivity. Then these manifolds are given by a system of equations of the form $J(x, u) = C$, where $J(x, u)$ are invariants of H, and C are constants.

For every H-solution $u = \varphi(x)$ there must exist such a set of invariants J that all equations $J(x, \varphi(x)) = C$ are satisfied identically with respect to the independent variables x. It follows that this solution also satisfies all the equations

$$D_i J(x, u) = 0 \qquad (i = 1, 2, \dots, n - m). \tag{P}$$

Therefore the general method for finding H-solutions consists in constructing such a set J of invariants of the group H for which all equations (P) are compatible with the equations (S). The compatibility conditions then form the system (S/H). Thus the latter is not determined only by the system (S) and the subgroup H, but also depends on the set J.

It follows from this that, in order to find all invariant-group solutions of the system (S), one must enumerate all possible subgroups H of the principal group and, for each of them, construct general typical sets of invariants. The realization of this program is connected with the group classification of all invariant-group solutions of the given system. In doing so it is useful to bear in mind that if H' and H'' are two subgroups and $H' \subset H''$, then any H''-solution is at the same time an H'-solution.

Of particular interest is the case when a complete set $J^\nu(x,u)$ $(\nu = 1,2,\dots,N)$ of independent invariants of the group H, with number $N > m$ (always $N < n$), can be chosen so that the first m of these invariants, for any fixed values of x, are uniquely invertible with respect to the variables u, while the remaining $N-m$ invariants do not contain u. In this case one can

find an entire class of H-solutions associated with a set of invariants of the form

$$J^k = \vartheta^k(J^{m+1},\dots,J^N) \quad (k = 1,2,\dots,m),$$

where ϑ are certain unknown functions. It is easy to verify that here the system (S/H) will be a quasilinear system with respect to the sought functions ϑ of the independent variables $y^j = J^{m+j}(x)$ $(j = 1,2,\dots,N-m)$. Indeed, if $u^k = U^k(x,\vartheta)$ $(k = 1,2,\dots,m)$ is the inversion of the equations $J^k(x,u) = \vartheta^k$ $(k = 1,2,\dots,m)$ with respect to u, then substitution into (S) of the expressions $u = U(x,\vartheta(y))$ transforms (S) into a quasilinear system whose coefficients depend only on the values of the variables y. The latter follows from the fact that the transformations H do not change the values of the quantities y and ϑ, and do change the values of the quantities x.

Every solution $\vartheta = \psi(y)$ of the obtained system (S/H) gives a certain H-solution of the system (S) of the form $u = U(x,\psi(y))$.

In conclusion it is appropriate to note that the new concept of invariant-group solutions introduced here includes the widely known and very useful, but narrower in scope, concept of the so-called "self-similar" solutions. This latter concept apparently first arose in the study of solutions of hydrodynamic equations of type (S_1), and is usually associated with dimensional analysis and symmetry considerations.

It is easy to see that for the system (S_1) the "self-similar" solutions are nothing other than invariant-group solutions for a one-parameter group of similarity transformations (more precisely, "generalized similarity"), which is determined by the operator $aX_3 + bX_5 + cX_6$ with arbitrary constants a, b, c.

At the same time, the old methods mentioned above leave a noticeable gap in the study of sets of particular solutions. For example, by these methods it would be difficult to obtain the following particular solution of the system (S_1) for $\gamma = 3$, corresponding to the one-parameter group H_1 with operator (where a, b are arbitrary constants) $aX_5 + a(b-3)X_6 + X_7$ and the complete set of independent invariants

$$J^1 = tu - \frac{tx}{t+a}, \quad J^2 = x^b t^{1-b} \rho, \quad J^3 = x^b t^{3-b} p,$$

$$J^4 = \frac{x}{t+a}.$$

Here the particular case considered above is realized, and the solution has the form

$$u = y + \frac{1}{t} U(y); \quad \rho = x^{-b} t^{b-1} R(y); \quad p = x^{-b} t^{b-3} P(y); \quad y = \frac{x}{t+a},$$

where the functions U, R, P satisfy the system (S_1/H_1) of ordinary differential equations obtained by direct substitution of these expressions for u, ρ, p into the system (S_1).

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Note: Figure translations are in progress. See original paper for figures.