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Abstract

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ON HOLOMORPHIC OPERATOR-FUNCTIONS

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Let G be an open connected domain of the complex plane and let A_λ be an operator-function holomorphic in the domain G,* whose values are linear closed operators acting from one complex Banach space $\mathfrak{B}_1$ into another $\mathfrak{B}_2$. Let, further, λ_0 be an arbitrary point of the domain G, and suppose that for $|\lambda-\lambda_0| < \rho$ ($\rho > 0$) the operator-function A_λ admits an expansion in an operator-norm convergent series:

$$A_\lambda = A_{\lambda_0} + \sum_{i=1}^{\infty} (\lambda - \lambda_0)^i C_i. \tag{1}$$

We note that the convergence of this series for $|\lambda - \lambda_0| < \rho$ is equivalent to the fact that for every positive number ρ_1 less than ρ there is a number M such that

$$|C_i| < M\rho_1^{-i} \quad (i = 1, 2, \ldots). \tag{2}$$

Consider an arbitrary vector $x_0 \in \mathfrak{Z}(A_{\lambda_0})$. By $\mu(x_0, A_{\lambda_0})$ we denote the largest of all nonnegative integers μ, for each of which there exist vectors $x_{\mu 0} = x_0, x_{\mu 1}, x_{\mu 2}, \ldots, x_{\mu\mu}$ such that

$$\sum_{i=0}^{k} C_i x_{\mu, k-i} = 0 \quad (k = 0, 1, \ldots, \mu),$$

where $C_0 = A_{\lambda_0}$. If among such numbers there is no largest one, then we put $\mu(x_0, A_{\lambda_0}) = \infty$. The linear set consisting of all vectors $x \in \mathfrak{Z}(A_{\lambda_0})$ for which $\mu(x, A_{\lambda_0}) = \infty$ will be denoted by $\mathfrak{N}(A_{\lambda_0})$, and the dimension of this linear set by $n(A_{\lambda_0})$.

Put, further, $k(\lambda) = \sup_y \inf_x |x|$, where the infimum is taken over all x solving the equation $A_\lambda x = y$, and the supremum over all $y \in \mathfrak{R}(A_\lambda)$, $|y| = 1$.

In 1954 I. Ts. Gokhberg suggested to the author that Theorem 1 of ([2]) (see also Theorem 3.6 of ([1])) be generalized to the case of a holomorphic operator-function whose values are Φ_+-operators. In the present note the indicated theorems are generalized and some other results are obtained, related to the same circle of questions ([1]), Theorem 8.2; ([3]), Theorem 12; ([4,5]).

§ 1. **Theorem 1.** *Suppose that for every point $\lambda \in G$ the operator A_λ is a Φ_+- (or Φ_-) operator. Then there exists a set $\Gamma \subset G$ such that $G - \Gamma$ is isolated in G and such that for all $\lambda \in \Gamma$ the function $\alpha(A_\lambda)$ has the constant value: $\alpha(A_\lambda) = \alpha_0$. If $\lambda \in G - \Gamma$, then $\alpha(A_\lambda) > \alpha_0$. Moreover, for all $\lambda \in G$ the function $n(A_\lambda)$ has the same constant value: $n(A_\lambda) = \alpha_0$.*

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* The terminology and notation are borrowed by us from ([1]). For convenience, instead of $\mathfrak{Z}_A$ we shall write $\mathfrak{Z}(A)$, etc.

Proof. First of all, let us note that it is enough to prove the following assertion: if λ_0 is an arbitrary point of G, then there exists a positive number r such that, for $0 < |\lambda - \lambda_0| < r$, the equality

$$\alpha(A_\lambda) = n(A_\lambda) = n(A_{\lambda_0})$$

holds. Indeed, from this assertion, by means of the usual arguments (see, for example, the proof of Theorem 3.3 in ([1])), Theorem 1 is easily obtained.

We proceed to the proof of the assertion formulated above. First consider the case when

$$\mathfrak{N}(A_{\lambda_0}) = \mathfrak{Z}(A_{\lambda_0}).$$

Let

$$x_{01}, x_{02}, \dots, x_{0\alpha} \quad (\alpha = \alpha(A_{\lambda_0}))$$

be a normalized basis in $\mathfrak{Z}(A_{\lambda_0})$. For each vector x_{0k} $(k = 1, 2, \dots, \alpha)$ there is a sequence $x_{1k}, x_{2k}, \dots$ such that

$$\sum_{i=0}^{j} C_i x_{j-i,k} = 0 \qquad (j = 0, 1, \dots). \tag{3}$$

The vectors x_{ik} $(k = 1, 2, \dots, \alpha;\ i = 1, 2, \dots)$ may be chosen so that the series

$$x_k(\lambda) = \sum_{i=0}^{\infty} (\lambda - \lambda_0)^i x_{ik} \qquad (k = 1, 2, \dots, \alpha) \tag{4}$$

converge for $|\lambda - \lambda_0| < r_1$ $(r_1 > 0)$. For this it is sufficient, for example, to ensure that the inequalities

$$|x_{ik}| \le k(\lambda_0) |C_0 x_{ik}| \qquad (k = 1, 2, \dots, \alpha;\ i = 1, 2, \dots) \tag{5}$$

are satisfied. In fact, from relations (2), (3), and (5) it is not difficult to obtain that

$$|x_{ik}| \le \left(Mk(\lambda_0) + 1\right)^i \rho_1^{-i} \qquad (k = 1, 2, \dots, \alpha;\ i = 0, 1, \dots),$$

whence follows the convergence of the series (4) for

$$|\lambda - \lambda_0| < \rho_1 (Mk(\lambda_0) + 1)^{-1}.$$

Further, from the equalities (3) it follows that

$$A_\lambda x_k(\lambda) = 0 \qquad (k = 1, 2, \dots, \alpha).$$

It is not difficult to establish the existence of such a positive number r_2 ($\leq r_1$) that, for $|\lambda - \lambda_0| < r_2$, the vectors $x_k(\lambda)$ ($k = 1, 2, \dots, \alpha$) are linearly independent and, consequently,

$$\alpha(A_\lambda) \geq \alpha = \alpha(A_{\lambda_0}).$$

But, on the other hand, there is a number r_3 (> 0) such that, for $|\lambda - \lambda_0| < r_3$, the reverse inequality

$$\alpha(A_\lambda) \leq \alpha(A_{\lambda_0})$$

holds ([1]), Theorem 7.1. Consequently,

$$\alpha(A_\lambda) = \alpha(A_{\lambda_0}) = n(A_{\lambda_0})$$

for

$$|\lambda - \lambda_0| < r = \min(r_2, r_3).$$

Now consider the case when

$$n(A_{\lambda_0}) < \alpha(A_{\lambda_0}).$$

Denote by $\mathfrak{N}_k$ the subspace of $\mathfrak{Z}(A_{\lambda_0})$ consisting of those vectors x for which

$$\mu(x, A_{\lambda_0}) \geq k.$$

By virtue of the finite-dimensionality of $\mathfrak{Z}(A_{\lambda_0})$, there is a natural number m such that

$$\mathfrak{N}_m = \mathfrak{N}(A_{\lambda_0}).$$

Denote by $\mathfrak{M}_k$ the direct complement to the subspace $\mathfrak{N}_{k+1}$ in the subspace $\mathfrak{N}_k$ ($k = 0, 1, \dots, m-1$). Let

$$x_{01}, x_{02}, \dots, x_{0n}, \; y_{01}, y_{02}, \dots, y_{0,\alpha-n} \quad (n = n(A_{\lambda_0}), \; \alpha = \alpha(A_{\lambda_0}))$$

be a normalized basis of $\mathfrak{Z}(A_{\lambda_0})$, composed of a basis

$$x_{01}, x_{02}, \dots, x_{0n}$$

of the subspace $\mathfrak{N}(A_{\lambda_0})$ and bases of the subspaces

$$\mathfrak{M}_k \qquad (k = 0, 1, \dots, m-1).$$

Denote $\mu(y_{0t}, A_{\lambda_0})$ by μ_t and choose elements

$$y_{1t}, y_{2t}, \dots, y_{\mu_t t}$$

such that

$$\sum_{i=0}^{l} C_i y_{l-i,t} = 0 \qquad (l = 0, 1, \dots, \mu_t;\ t = 1, 2, \dots, \alpha - n). \tag{6}$$

Next, let $\mathfrak{K}$ be some $(\alpha - n)$-dimensional normed space, and let

$$z_1, z_2, \dots, z_{\alpha-n}$$

be a normalized basis in $\mathfrak{K}$. Denote by $\mathfrak{B}_1$ the direct sum of the spaces $\mathfrak{B}_1$ and $\mathfrak{K}$, in which the norm is defined by the equality

$$|y + z| = |y| + |z| \qquad (y \in \mathfrak{B}_1,\ z \in \mathfrak{K}).$$

Denote by $\widetilde{C}_i$ $(i = 0, 1, \dots)$ the operators acting from $\widetilde{\mathfrak{B}}_1$ into $\mathfrak{B}_2$, coinciding in $\mathfrak{B}_1$ with C_i and defined on $\mathfrak{N}$ by the equalities

$$\widetilde{C}_i z_t = -\sum_{j=1}^{\mu_t+1} C_{i+j} y_{\mu_t+1-j,t} \qquad (t = 1, 2, \dots, \alpha - n;\ i = 0, 1, \dots). \tag{7}$$

It is not hard to show that $\widetilde{C}_0$ vanishes on $\mathfrak{N}$ only at zero; hence it follows that $\mathfrak{z}(\widetilde{C}_0) = \mathfrak{z}(C_0)$. Next put

$$\widetilde{A}_\lambda = \widetilde{C}_0 + \sum_{i=1}^{\infty} (\lambda - \lambda_0)^i \widetilde{C}_i.$$

It is not hard to establish that this series, together with the series (1), converges for $|\lambda - \lambda_0| < \rho$ and that, for the indicated values of λ, the operator $\widetilde{A}_\lambda$, being an extension by $\alpha - n$ dimensions of the operator A_λ, is likewise a Φ_+- (or Φ_-) operator.

We now show that $\mathfrak{N}(\widetilde{A}_{\lambda_0}) = \mathfrak{Z}(\widetilde{A}_{\lambda_0})$. For this it is enough to establish that for every vector y_{0t} $(t = 1, 2, \dots, \alpha - n)$ there exists a sequence $\tilde{y}_{1t}, \tilde{y}_{2t}, \dots$ such that

$$\sum_{i=0}^{j-1} \widetilde{C}_i \tilde{y}_{j-i,t} + \widetilde{C}_j y_{0t} = 0 \qquad (j = 0, 1, \dots). \tag{8}$$

But it suffices to put $\tilde{y}_{it} = y_{it}$ $(i = 1, 2, \dots, \mu_t)$; $\tilde{y}_{\mu_t+1,t} = z_t$; $\tilde{y}_{it} = 0$ $(i = \mu_t + 2, \mu_t + 3, \dots)$, and, by virtue of equalities (6) and (7), relations (8) will be fulfilled. Applying to the operator $\widetilde{A}_\lambda$ the result of the first part of the proof, we obtain that for $|\lambda-\lambda_0|<r$ the equality $\alpha(\widetilde{A}_\lambda)=\alpha(\widetilde{A}_{\lambda_0})=\alpha(A_{\lambda_0})$ holds, and the basis of the subspace $\mathfrak{Z}(A_\lambda)$ consists of the vectors

$$x_k(\lambda)=\sum_{i=0}^{\infty}(\lambda-\lambda_0)^i x_{ik}\quad(k=1,2,\dots,n);\qquad y_t(\lambda)=\sum_{i=0}^{\mu_t}(\lambda-\lambda_0)^i y_{it}+(\lambda-\lambda_0)^{\mu_t+1}z_t$$

$$(t=1,2,\dots,\alpha-n).$$

Since $\mathfrak{Z}(A_\lambda)=\mathfrak{Z}(\widetilde{A}_\lambda)\cap\mathfrak{B}_1$, we have $\dim\mathfrak{Z}(\widetilde{A}_\lambda)/\mathfrak{Z}(A_\lambda)\le\dim\widetilde{\mathfrak{B}}_1/\mathfrak{B}_1=\alpha-n$. On the other hand, it is easy to see that for $\lambda\neq\lambda_0$ the subspace spanned by the elements $y_t(\lambda)$ $(t=1,2,\dots,\alpha-n)$ intersects $\mathfrak{B}_1$ only at zero, and, consequently, $\dim\mathfrak{Z}(\widetilde{A}_\lambda)/\mathfrak{Z}(A_\lambda)\ge\alpha-n$.

Thus, $\dim\mathfrak{Z}(\widetilde{A}_\lambda)/\mathfrak{Z}(A_\lambda)=\alpha-n$ for $0<|\lambda-\lambda_0|<r$, and therefore $\alpha(A_\lambda)=\dim\mathfrak{Z}(A_\lambda)=\alpha-(\alpha-n)=n(A_{\lambda_0})$.

It remains for us to prove that $n(A_\lambda)=n(A_{\lambda_0})$ for $|\lambda-\lambda_0|<r$. Suppose that for some point λ' $(|\lambda'-\lambda_0|<r)$ $n(A_{\lambda'})<n(A_{\lambda_0})$. But then there exists a number $r'(>0)$ such that $\alpha(A_\lambda)=n(A_{\lambda'})<n(A_{\lambda_0})$ for $0<|\lambda-\lambda'|<r'$. Since there are points simultaneously satisfying the inequalities $0<|\lambda-\lambda_0|<r$ and $0<|\lambda-\lambda'|<r'$, we have arrived at a contradiction. The theorem is completely proved.

Theorem 2. *Suppose that for every point $\lambda\in G$ the operator A_λ is a Φ_--operator. Then there exists a set $\Gamma\subset G$ such that $G-\Gamma$ is isolated in G and such that for all $\lambda\in\Gamma$ the function $\beta(A_\lambda)$ has the constant value: $\beta(A_\lambda)=\beta_0$. If, however, $\lambda\in G-\Gamma$, then $\beta(A_\lambda)>\beta_0$. Moreover, $\mathfrak{R}(A_\lambda)=\mathfrak{Z}(A_\lambda)$ for all $\lambda\in\Gamma$.*

Proof. Let $\mathfrak{D}=\mathfrak{D}(A_\lambda)$ for $\lambda\in G$. Denote by $\hat{A}_\lambda$ the operator acting in the same way as A_λ, from the space $\widehat{\mathfrak{B}}=\overline{\mathfrak{D}}$ into the space $\mathfrak{B}_2$, and let $\hat{A}_\lambda^+$ be the operator adjoint to it. Applying

Theorem 1 to the holomorphic operator-function $\hat{A}_\lambda^+$, whose values are Φ_+-operators, and taking into account that $\alpha(\hat{A}_\lambda^+)=\beta(A_\lambda)$ and that from the equality $\mathfrak{R}(\hat{A}_\lambda^+)=\mathfrak{Z}(\hat{A}_\lambda^+)$ there follows the equality $\mathfrak{R}(A_\lambda)=\mathfrak{Z}(A_\lambda)$, we immediately obtain Theorem 2.

§ 2. Let Γ be the set of complex numbers referred to in the formulation of Theorem 1 or Theorem 2. It can be proved that the function $k(\lambda)$ is continuous on Γ. With the aid of this assertion and the method of proof from ([5]), the results of § 2 of the note ([4]) carry over completely to the case under consideration.

Let us note in conclusion that, by means of the general device indicated by B. Sz.-Nagy ([6,7]) (see also ([1])), all the results of the present note can be carried over to the case where the holomorphy condition on A_λ is replaced by the following more general condition: the values of the operator-function A_λ are linear closed operators having, for all $\lambda \in G$, one and the same domain of definition $\mathfrak{D} = \mathfrak{D}(A_\lambda)$, and for each point $\lambda_0 \in G$ there exist a positive number ρ and linear operators $C_1, C_2, \ldots$ such that $\mathfrak{D}(C_i) \supseteq \mathfrak{D}$ $(i = 1, 2, \ldots)$, and such that, for $|\lambda - \lambda_0| < \rho$, for every $x \in \mathfrak{D}$,

$$A_\lambda x = A_{\lambda_0} x + \sum_{i=1}^{\infty} (\lambda - \lambda_0)^i C_i x.$$

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Note: Figure translations are in progress. See original paper for figures.