Invariants of Homogeneous and Isotropic Turbulence in a Compressible Viscous Fluid

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Abstract

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Invariants of Homogeneous and Isotropic Turbulence in a Compressible Viscous Fluid

(Presented by Academician A. N. Kolmogorov, 29 IV 1958)

1. Derivation of the invariants. The equations expressing the laws of conservation of momentum, mass, and energy in hydrodynamics have the following form:

$$\frac{\partial \rho v_i}{\partial t} = -\sum_{k=1}^{3} \frac{\partial \pi_{ik}}{\partial x_k}; \tag{1}$$

$$\frac{\partial \rho}{\partial t} = -\sum_{k=1}^{3} \frac{\partial \rho v_k}{\partial x_k}; \tag{2}$$

$$\frac{\partial \rho U}{\partial t} = -\sum_{k=1}^{3} \frac{\partial E_k}{\partial x_k}. \tag{3}$$

Here ρ is the density of the fluid; v_i are the components of its velocity; π_{ik} is the tensor of the momentum-flux density, which for a viscous compressible fluid has the form

$$\pi_{ik} = p\delta_{ik} + \rho v_i v_k - \eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik} \sum_{l=1}^{3} \frac{\partial v_l}{\partial x_l} \right) - \zeta \delta_{ik} \sum_{l=1}^{3} \frac{\partial v_l}{\partial x_l}, \tag{4}$$

where p is the pressure; η and ζ are the viscosity coefficients; U is the total energy of a unit mass of the fluid; E_k are the components of the energy-flux-density vector*.

We shall regard all hydrodynamic elements of the flow as random variables (1), satisfying the conditions of homogeneity and isotropy. Averages will be denoted by an overbar.

Taking averages of equations (2) and (3), we see that, owing to homogeneity, the averages $\bar{\rho}$ and $\overline{\rho U}$ do not change with time. Therefore in these equations the quantities ρ and ρU may be replaced by their fluctuations $\rho' = \rho - \bar{\rho}$ and $(\rho U)' = \rho U - \overline{\rho U}$.

Take, from among the quantities ρv_i $(i = 1, 2, 3)$, ρ', and $(\rho U)'$, any two, which we denote by α and β, and denote the components of the vectors of whose divergences they are, according to equations (1)–(3), by A_k and B_k. Consider the second correlation moment

$$\overline{\alpha(x_1, x_2, x_3, t)\,\beta(x_1^*, x_2^*, x_3^*, t)} = \overline{\alpha\beta^*}(r, t),$$

where $\mathbf{r}$ is the vector having components $\xi_k = x_k^* - x_k$. Using the homogeneity of the flows, we obtain

$$\frac{\partial \overline{\alpha\beta^*}}{\partial t} = \overline{\alpha \sum_{k=1}^{3} \frac{\partial B_k^*}{\partial x_k^*} + \sum_{k=1}^{3} \frac{\partial A_k}{\partial x_k}\,\beta^*} = \sum_{k=1}^{3} \frac{\partial}{\partial \xi_k}\left(\overline{\alpha B_k^*} - \overline{A_k\beta^*}\right).$$

* The form of π_{ik}, U, and E_k is immaterial for the present paragraph; it will be needed in § 3.

We integrate both sides of this equality over the volume of the sphere $r = |\mathbf{r}| \leq R$ and replace, on the right-hand side, the integral of the divergence by a surface integral. Assuming that αB_k^* and $A_k\beta^*$ are $o(1/r^2)$, we shall have, as $R \to \infty$, that the right-hand side tends to zero, and therefore

$$\lim_{R\to\infty} \frac{\partial}{\partial t} \iiint_{r\leq R} \overline{\alpha\beta^*(\mathbf{r}, t)}\, dV_r = 0.$$

Consequently,

$$\iiint \overline{\alpha\beta^*(\mathbf{r}, t)}\, dV_r$$

does not depend on time, i.e., is an invariant.

If the flow is not only homogeneous but also isotropic, then among the invariants obtained there are only 4 different ones that are nonzero. These invariants have the following mechanical meaning:

$$\Lambda_1 = \iiint \overline{\rho\mathbf{v} \cdot \rho\mathbf{v}^*}\, dV_r = \lim_{V\to\infty} \frac{1}{V} \overline{\left(\iiint_V \rho\mathbf{v}\, dV_x\right)^2},$$
$$\Lambda_2 = \iiint \overline{\rho' \cdot \rho o'^*}\, dV_r = \lim_{V\to\infty} \frac{1}{V} \overline{\left(\iiint_V \rho'\, dV_x\right)^2},$$
$$\Lambda_3 = \iiint \overline{(\rho U)'(\rho U)'^*}\, dV_r = \lim_{V\to\infty} \frac{1}{V} \overline{\left(\iiint_V (\rho U)'\, dV_x\right)^2},$$
$$\Lambda_4 = \iiint \overline{\rho'(\rho U)'^*}\, dV_r = \lim_{V\to\infty} \frac{1}{V} \overline{\left(\iiint_V \rho'\, dV_x\right)\left(\iiint_V (\rho U)'\, dV_x\right)}.$$

To prove, for example, the first of these equalities, we write the square of the momentum of the volume V in the form of a double integral

$$\overline{\left(\int \rho\mathbf{v}\, dV\right)^2} = \iint \overline{\rho\mathbf{v} \cdot \rho\mathbf{v}^*}\, dV\, dV^* = \Lambda_1 V + o(V),$$

from which our assertion follows.

2. Example of a flow with invariants different from zero.
We scatter points in three-dimensional space according to Poisson' s law, so that the probability that a region of volume V contains m points is

$$\frac{V^m}{m!} e^{-V}.$$

Around each point we describe a sphere of unit volume, inside which we set $|\rho\mathbf{v}|$, ρ, and ρU equal to products $\varphi(r)\xi$, where $\varphi(r)$ is a sufficiently smooth function of the distance from the center of the sphere, vanishing together with its derivatives at $r = \sqrt[3]{3/4\pi}$ (i.e., on the boundary of the sphere) and increasing monotonically as r decreases, while ξ is a random variable. We take the direction of the velocity to be the same at all points of the sphere and uniformly distributed over all directions. Here $\varphi(r)$ and ξ are chosen so that the spheres possess, independently of one another: 1) momenta,* whose squares are equal to α; 2) masses ρ with mean values $\overline{\rho}$ and variances β; 3) energies ρU with mean values $\overline{\rho U}$ and variances γ; 4) correlation coefficients δ between ρ' and $(\rho U)'$.

We take all space filled with an ideal gas, which at each point has momentum, density, and total energy

$$\rho U = \rho \frac{v^2}{2} + \rho c_{vT}$$

(where c_v is the heat capacity at constant volume, T is the temperature), equal to the sums of these quantities for those spheres to which this point belongs.

* By the momentum of a sphere we mean the integral over the volume of the sphere $\int \rho\mathbf{v}\, dV$.

We shall prove that, for homogeneous and isotropic turbulence specified in this way at the initial instant of time, the values of the invariants are respectively α, β, γ, and $\sqrt{\beta\gamma}\,\delta$. Indeed, for example, for the first of them

$$\overline{\left(\int_V \rho v\, dV\right)^2} = \sum_{m=0}^{\infty} \frac{V^m}{m!} e^{-V} m\alpha + o(V) = \alpha V + o(V),$$

whence it follows that $\Lambda_1 = \alpha$.

Remark. In deriving the invariants we assumed that the correlations between hydrodynamic quantities at any instant of time decrease sufficiently rapidly at infinity, namely as $o(1/r^2)$. Since disturbances in a compressible fluid propagate with finite velocity, it seems plausible that from the fulfillment of this condition at the initial instant of time its validity follows at any subsequent instant of time. For the same reason, in the case when the fluid possesses compressibility, however small, the objections that are made in the case of an incompressible fluid concerning Loitsyansky' s invariant are inapplicable (see ([2])).

If the fluid is incompressible, then all the invariants, except Λ_3, are equal to zero. The invariant Λ_2 was previously obtained by Chandrasekhar ([3]).

3. The law of decay of homogeneous and isotropic turbulence at the final stage in an ideal gas possessing viscosity and thermal conductivity. We shall prove that, at the final stage of turbulence decay, when the nonlinear terms in the equations of motion may be neglected, the equalities

$$\overline{v(t)^2} = \left(\frac{\Lambda_1}{48\rho^2} + \frac{a^2}{16\rho^2}\Lambda_2\right)\left(\frac{1}{\pi[\nu' + (\chi-1)\lambda]t}\right)^{3/2} + \frac{\Lambda_1}{12\rho^2}\left(\frac{1}{2\pi\nu t}\right)^{3/2} + o\left(\frac{1}{t^{3/2}}\right), \quad (5)$$

$$\overline{\rho'(t)^2} = \left(\frac{\Lambda_1}{48a^2} + \frac{\Lambda_2}{16}\right)\left(\frac{1}{\pi[\nu' + (\chi-1)\lambda]t}\right)^{3/2} + o\left(\frac{1}{t^{3/2}}\right),$$

$$\overline{T'(t)^2} = \frac{\Lambda_3}{8\rho^2}\left(\frac{1}{2\pi\lambda t}\right)^{3/2} + o\left(\frac{1}{t^{3/2}}\right),$$

$$\overline{\rho' T'(t)} = o\left(\frac{1}{t^{3/2}}\right).$$

For the proof, for example, of formula (5), we take the spectral expansion

$$\overline{v(t)^2} = \iiint [f_{dd}(p,t) + 2f_{nn}(p,t)]\, dV, \tag{6}$$

where

$$f_{nn}(p,t) = f_{nn}(p,0)e^{-2\nu p^2 t}$$

corresponds to the incompressible component, and $f_{dd}(p,t)$ to randomly distributed acoustic waves (see ([4])):

$$f_{dd}(p,t) = \left|c_1(p)e^{k_1 t} + c_2(p)e^{k_2 t} + c_3(p)e^{k_3 t}\right|^2,$$

where $c(p)$ are random quantities; $c_3(p) \to 0$ as $p \to 0$;

$$\begin{aligned} k_1 &= -\frac{p^2}{2}[\nu' + (\chi - 1)\lambda] + O(p^3) - ip[a + O(p)], \\ k_2 &= -\frac{p^2}{2}[\nu' + (\chi - 1)\lambda] + O(p^3) + ip[a + O(p)], \\ k_3 &= -p^2\lambda + O(p^3), \end{aligned} \tag{7}$$

$$\nu = \frac{\eta}{\rho}; \qquad \nu' = \frac{4\eta}{3\rho} + \frac{\zeta}{\rho};$$

λ is the coefficient of thermal diffusivity; χ is the ratio of heat capacities; a is the speed of sound.

It is proved that, for any $p > \varepsilon > 0$, the real parts of all k are less than $c < 0$.* Therefore, by making an error of infinitely small higher order, one may reduce the domain of integration in (6) to an arbitrarily small neighborhood of the origin of coordinates and take, in the integrand for k, the principal values from (7), while regarding the coefficients as constant and equal to their values at zero. When integrating such an integrand we may extend the domain of integration to the whole space, since this only adds a term that decreases exponentially with time. Thus:

$$\overline{v(t)^2} = \iiint \left[\overline{|c_1(0)|^2} + \overline{|c_2(0)|^2}\right] e^{-p^2[\nu' + (x-1)\lambda]t}\, dV +$$

$$+ \iiint 2f_{nn}(0,0)e^{-2\nu p^2 t}\, dV +$$

$$+ \iiint 2\left[\mathrm{Re}\, \overline{c_1(0)c_2(0)^*} \cos 2apt - \mathrm{Im}\, \overline{c_1(0)c_2(0)^*} \sin 2apt\right] e^{-p^2[\nu' + (x-1)\lambda]t}\, dV +$$

$$+\,o\left(\frac{1}{t^{5/2}}\right), \tag{8}$$

where the asterisk denotes the complex conjugate quantity.

At the last stage of the decay of turbulence, when the density fluctuations ρ' are sufficiently small in comparison with the mean density $\bar{\rho}$, the value of the invariant Λ_1 will differ arbitrarily little from the quantity

$$\Lambda_1' = \bar{\rho}^2 \iiint \overline{\mathbf{v}\cdot\mathbf{v}^*}\,dV = 24\pi^3\bar{\rho}^2 f_{dd}(0,t) = 24\pi^3\bar{\rho}^2 f_{nn}(0,t) =$$

$$= 24\pi^3\bar{\rho}^2\,\overline{|c_1(0)+c_2(0)|^2} =$$

$$= 24\pi^3\bar{\rho}^2\left[\overline{|c_1(0)|^2} + \overline{|c_2(0)|^2} + \overline{c_1(0)c_2(0)^*} + \overline{c_2(0)c_1(0)^*}\right],$$

which is an invariant for the linearized flow. From the relations holding between the complex amplitudes of plane waves of velocity and density fluctuation, we obtain:

$$\Lambda_2 = \iiint \overline{\rho'\rho o'^*}\,dV = 8\pi^3\frac{\bar{\rho}^2}{a^2}\overline{|c_1(0)-c_2(0)|^2}.$$

Consequently,

$$\overline{|c_1(0)|^2} + \overline{|c_2(0)|^2} = \frac{\Lambda_1'}{48\pi^3\bar{\rho}^2} + \frac{a^2}{16\pi^3\bar{\rho}^2}\Lambda_2.$$

Taking this relation into account and evaluating the first two integrals in (8), we obtain formula (6); the third integral in (8) is $o(1/t^{5/2})$.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.