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Abstract

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CHEBYSHEV SETS IN BANACH SPACES

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No. 1. A set M lying in a metric space R will be called **Chebyshev** if for every point $x \in R$ there exists a unique point $y \in M$ such that the distance $\rho(x, y) = \rho(x, M)$ (in this connection see ([1])).

In this note we establish the **convexity of compact Chebyshev sets in Banach spaces under certain restrictions on the unit sphere**, namely, in spaces with a smooth unit sphere having uniformly small skewness (see No. 6). In particular, this thereby establishes the convexity of compact Chebyshev sets in Hilbert space.

No. 2. In what follows X denotes a real Banach space; $E_a(p)$ denotes the open ball of radius a with center $p \in X$; and $O_a(p)$ denotes the complement of $E_a(p)$ in X.

Let a set $M \subset X$ and a plane $L \subset X$ be given. Denote by (L, M, a) the intersection of L and all $O_a(p)$, $p \in L$, that contain M. Then:

1) (L, M, a) is closed (as the intersection of closed sets).
2) If $M \subseteq M'$, then $(L, M, a) \subseteq (L, M', a)$.
3) If $a_1 < a_2$, then $(L, M, a_1) \subseteq (L, M, a_2)$.

Property 2) follows immediately from the definition. Property 3) is also seen directly; indeed, if $x \in L$ does not belong to (L, M, a_2), i.e. lies inside some ball $E_{a_2}(p_2)$, $p_2 \in L$, which does not contain M, then x lies inside a ball $E_{a_1}(p_1)$, $p_1 \in L$, which is contained in the ball $E_{a_2}(p_2)$. Thus the complement of (L, M, a_1) in L contains the complement of (L, M, a_2); hence $(L, M, a_1) \subseteq (L, M, a_2)$.

No. 3. In the case when $L \equiv X$, we shall denote (L, M, a) by (M, a) and call this set the **a-envelope of the set** M. On the other hand, when $L \neq X$, we shall often consider sets inside L, abstracting from X; in such cases we shall call L a **space**, considering that some point of L has been taken as the origin.

The following properties of a-envelopes hold:

4) $M \subseteq (M, a)$ for every a.
5) (L, M, a_0) coincides with its a-envelope in the space L for every $a \leq a_0$ (proved analogously to 3), taking into account 4)).

6) The closure of the set-theoretic sum of all (L, M, a), $0 < a < \infty$, coincides with its a-envelope in the space L for every a (proved analogously to 5)).

No. 4. Consider an n-dimensional Banach space X_n with a smooth unit sphere, i.e. one having at each point of the unit sphere a unique supporting hyperplane.

Lemma 1. Let K be the set of vertices of a full-dimensional simplex $S \subset X_n$; then the closure of the set-theoretic sum of all a-envelopes of K is the simplex S.

Lemma 2. Let $M \subset X_n$; if $M \equiv (M, a)$ for every a, then M is either convex or degenerate (i.e. lies in some hyperplane).

Lemma 2 follows from Lemma 1.

No. 5. Here we again consider an infinite-dimensional Banach space X. We call a closed ball $\bar{E}$ a supporting ball to a certain set Q at the points of its subset Q_1, if inside $\bar{E}$ there are no points of Q, and Q_1 lies on the boundary of $\bar{E}$.

Lemma 3. *If $M \subset X$ is compact, then for any boundary point y_0 of the set (L, M, a), regarded as a set of the space L, there exists a closed ball $\bar{E}_a(p)$, $p \in L$, supporting (L, M, a) at the point y_0 and also supporting M.*

Lemma 4 (on removing a ball). *Let $M \subset X$ be a compact set; $\bar{E}$ a closed ball supporting M at the single point x_0; e a vector going from x_0 into the interior of the ball $\bar{E}$. Then there exists $\lambda_0 > 0$ such that the ball*

$$\bar{E}' = \bar{E} + \lambda e$$

does not intersect M for all λ, $0 < \lambda \leq \lambda_0$.

Lemma 5. *Let the ball $\bar{E}$ be smooth; A a supporting hyperplane to $\bar{E}$ at an arbitrary boundary point x; e a vector which, when applied at x, goes into that (open) half-space of X, relative to A, in which the ball $\bar{E}$ lies. Then e goes into the interior of the ball $\bar{E}$.*

Proof. Suppose that no point of the segment $z = x + te$, $0 \leq t \leq t_0$, is an interior point of $\bar{E}$. Denote by W the convex hull of this segment and $\bar{E}$. The closed set W is a convex body; $z = x + te$, $0 < t < t_0$, is its boundary point. By Mazur' s theorem ([2]), at the point z there exists a hyperplane B supporting W. This hyperplane contains the entire segment $z = x + te$, $0 \leq t \leq t_0$, and, consequently, is a supporting hyperplane to the ball $\bar{E}$ at the point x_0. Moreover, by construction, B does not coincide with A. We have arrived at a contradiction with the definition of smoothness of the ball $\bar{E}$. Hence it follows that there exists a point $z_1 = x + t_1 e$, $0 < t_1 \leq t_0$, interior to $\bar{E}$; consequently, the whole segment $z = x + te$, $0 < t \leq t_1$, consists of interior points of $\bar{E}$. The lemma is proved.

No. 6. Let X be a strictly convex Banach space; S its unit sphere; L_n an n-dimensional subspace; $x \in S$ an arbitrary point whose distance from L_n is $\leq \varepsilon < 1$; Λ_{n-1} an $(n-1)$-dimensional plane tangent to S at the point x and parallel to L_n. Consider the intersection of S and L_n; owing to the strict convexity of X there will be exactly two points $y_1, y_2 \in L_n$ at which the $(n-1)$-dimensional tangent plane to S is parallel to Λ_{n-1}; denote by y the one of them nearest to the point x; by $\rho(x, y)$ the distance between x and y. We shall say that X is a **space with uniformly small skewness** if there exists a number k $(k > 1)$ such that

$$\rho(x, y) \leq f(\varepsilon), \qquad f(\varepsilon) = k\varepsilon,$$

for all $x \in S$.

A space with uniformly small skewness has the following property, which we shall use below: if L_n is an n-dimensional plane passing through the center p of the sphere $S_a(p)$, and all the other notations retain the analogous meaning, then $\rho(x, y) \leq f(\varepsilon)$ for all a and $x \in S_a(p)$.

Remark. It can be shown that not every uniformly convex space is a space with uniformly small skewness; however, Hilbert space belongs to this type (for it $\rho(x, y) \leq \varepsilon\sqrt{2}$).

No. 7. Let M be a Chebyshev compact set in a space with uniformly small skewness and a smooth unit sphere. For any $\varepsilon > 0$, choose some ε-net of the set M, construct the smallest finite-dimensional plane L_ε passing through all points of this ε-net, and construct the set (L_ε, M, a).

Lemma 6. *For any $a > 0$, every point of the set (L_ε, M, a) is at distance $\leq f(\varepsilon)$ from M.*

Proof. From the definition of the set (L_ε, M, a) it follows that it is bounded and, consequently, has boundary points in L_ε. Consider any one of its boundary points y_0. According to Lemma 3 there exists a closed ball $\overline{E}_a(p)$, supporting the set (L_ε, M, a) at the point y_0 and at the same time supporting M. Since M is a Chebyshev set, $\overline{E}_a(p)$ intersects M in the single point x_0. We shall prove that $\rho(x_0, y_0) \leq f(\varepsilon)$. Denote by A and B the hyperplanes in X supporting $\overline{E}_a(p)$ respectively at the points x_0 and y_0. If $\rho(x_0, y_0) > f(\varepsilon)$, then the intersections A and B with L_ε are not parallel. Therefore there is a vector e which belongs to L_ε and satisfies the following two conditions: 1) being applied to some point of the hyperplane A, it is directed into that open half-space of X with respect to A in which the ball $\overline{E}_a(p)$ lies; 2) being applied to some point of the hyperplane B, it is directed into that open half-space of X with respect to B in which the ball $\overline{E}_a(p)$ does not lie. By Lemma 5 and the first of these two conditions, the vector e, with point of application x_0, is directed into the interior of the ball $\overline{E}_a(p)$. On the basis of Lemma 4 there exists a number $\lambda_0 > 0$ such that the ball

$$\overline{E}_a(p') = \overline{E}_a(p) + \lambda e, \qquad 0 < \lambda \leq \lambda_0,$$

does not intersect M, and moreover has its center p' in the plane L_ε. By virtue of the second condition in the definition of the vector e, for sufficiently small λ the point y_0 will lie inside the ball $\overline{E}_a(p')$, i.e. it will be an exterior point of the set (L_ε, M, a) in the space L_ε.

We have obtained a contradiction to the condition that y_0 is a boundary point of (L_ε, M, a). Thus the inequality $\rho(y_0, x_0) \le f(\varepsilon)$ has been proved by contradiction. Consequently, $\rho(y_0, M) \le f(\varepsilon)$.

Now we shall prove that every interior point of the set (L_ε, M, a) in the space L_ε is likewise at distance $\le f(\varepsilon)$ from M.

Suppose that inside (L_ε, M, a) in the space L_ε there is a point y for which $\rho(y, M) > f(\varepsilon)$. Since the set (L_ε, M, a) is compact, the continuous function $\rho(y, M)$ attains on it a maximum $m = \rho(q, M) > f(\varepsilon)$, $q \in L_\varepsilon$. The point q lies inside (L_ε, M, a), since, according to what has been proved, on the boundary we have $\rho(x, M) \le f(\varepsilon)$. Consider in X the ball $\overline{E}_m(q)$. This ball has a unique point of intersection with M, namely x_0. Since $\varepsilon \le f(\varepsilon) < m$, the supporting hyperplane C of the ball $\overline{E}_m(q)$ at the point x_0 is not parallel to L_ε. Therefore there exists a vector $e \in L_\varepsilon$ which, being applied to the point x_0, is directed into that half of the space X with respect to the hyperplane C in which the ball $\overline{E}_m(q)$ lies, i.e. into the interior of this ball. Again applying Lemma 4, we find that there exists $\lambda_0 > 0$ such that the ball

$$\overline{E}_m(q_1) = \overline{E}_m(q) + \lambda e, \qquad 0 < \lambda \le \lambda_0,$$

has no common points with M. For sufficiently small λ, the point q_1 will lie inside (L_ε, M, a) in the space L_ε, and moreover $\rho(q_1, M) > m$.

We have arrived at a contradiction to the definition of the number m. Thus it has been proved that for every point $y \in (L_\varepsilon, M, a)$ the relation $\rho(y, M) \le f(\varepsilon)$ holds, i.e. the lemma is proved.

Theorem. A Chebyshev compact set in a Banach space with uniformly small skewness and a smooth unit sphere is a convex set.

Proof. Let G_ε be the closure of the set-theoretic sum of all (L_ε, M, a), $0 < a < \infty$. According to No. 4,

$$G_\varepsilon = (G_\varepsilon, a) \text{ in } L_\varepsilon.$$

By the construction of L_ε, the set G_ε is nondegenerate in the space L_ε. Hence, by Lemma 2, G_ε is convex. By Lemma 6, the distance from any point $y \in G_\varepsilon$ to the set M satisfies the inequality

$$\rho(y, M) \le f(\varepsilon).$$

By the definition of the function $f(\varepsilon)$, we have $f(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Thus, the compact set M is approximated arbitrarily well by convex sets. It follows that M is convex. The theorem is proved.

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References

[1] N. V. Efimov, S. B. Stechkin, DAN, 118, No. 1, 17 (1958). [2] S. Mazur, Stud. Math., 4, 70 (1933).

Note: Figure translations are in progress. See original paper for figures.