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Abstract

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MATRIX ELEMENTS OF IRREDUCIBLE UNITARY REPRESENTATIONS OF THE GROUP OF MOTIONS OF LOBACHEVSKY SPACE AND GENERALIZED FOCK-MELER TRANSFORMS

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Integral formulas for the matrix elements of irreducible unitary representations of the group G of homogeneous linear transformations preserving the quadratic form $x_1^2+x_2^2+\dots+x_n^2-x_{n+1}^2$ (the group of motions of Lobachevsky space) were indicated by us in the paper (^{1}). In the case when these representations are realized in the space of functions on the sphere, one easily computes the matrix elements satisfying the functional equation $f(g)=f(g\omega)$, where ω is an element of the subgroup Ω_n of rotations of the subspace $x_{n+1}=0$ (associated spherical functions). These matrix elements have the form

$$\frac{2^p\Gamma(p+1)\Gamma(-p-i\mu+{}^1/_2)}{\Gamma(-p-i\mu-k+{}^1/_2)}\sqrt{\frac{\Gamma(2p+k)(p+k)}{k!\,p\Gamma(2p)}}\,\mathrm{sh}^{-p}\,\alpha\,P^{-p-k}_{-{}^1/_2+i\mu}(\mathrm{ch}\,\alpha), \tag{1}$$

where $p=(n-2)/2$; μ is the parameter specifying the representation; k is the parameter specifying the element (k is an integer). For $k=0$ we obtain zonal spherical functions satisfying the equation $f(\omega_1 g\omega_2)=f(g)$. A differential equation for these functions was indicated by M. G. Krein in the paper (^{2}). The integral representation leads to the relation

$$\frac{1}{\pi}\int_0^\pi C_k^p(\cos\varphi)\,(\cos\varphi\;\mathrm{sh}\,\alpha+\mathrm{ch}\,\alpha)^{-p-{}^1/_2+i\mu}\sin^{2p}\varphi\,d\varphi=$$

$$=\frac{\Gamma(2p+k)\Gamma(-p+{}^1/_2+i\mu)}{2^{p-1}k!\Gamma(p)\Gamma(-p-k+{}^1/_2+i\mu)}\,\mathrm{sh}^{-p}\,\alpha\,P^{-p-k}_{-{}^1/_2+i\mu}(\mathrm{ch}\,\alpha), \tag{2}$$

where $C_k^p(\cos\varphi)$ are Gegenbauer polynomials.

The usual considerations connected with the theory of group representations (see, for example, (^{3},^{4})) lead to the following addition theorem. Let

$$\operatorname{ch}\alpha = \operatorname{ch}\alpha_1 \operatorname{ch}\alpha_2 + \operatorname{sh}\alpha_1 \operatorname{sh}\alpha_2 \cos\varphi.$$

Then

$$\operatorname{sh}^{-p}\alpha\, P^{-p}_{-{}^1/_2+i\mu}(\operatorname{ch}\alpha) = \frac{2^p\Gamma(p+1)}{p}\sum_{k=0}^{\infty}(p+k)\prod_{r=0}^{k-1}\left[(p+r+{}^1/_2)^2+\mu^2\right]\times$$

$$\times \operatorname{sh}^{-p}\alpha_1 \operatorname{sh}^{-p}\alpha_2 P^{-p-k}_{-{}^1/_2+i\mu}(\operatorname{ch}\alpha_1) P^{-p-k}_{-{}^1/_2+i\mu}(\operatorname{ch}\alpha_2) C^p_k(\cos\varphi). \tag{3}$$

From formula (3) it follows that

$$\int_0^{\pi} \operatorname{sh}^{-p}\alpha\, P^{-p}_{-\frac{1}{2}+i\mu}(\operatorname{ch}\alpha)\, C^p_k(\cos\varphi)\, \sin^{2p}\varphi\, d\varphi =$$

$$= \frac{\pi\Gamma(2p+k)}{2^{p-1}k!\Gamma(p)}\prod_{r=0}^{k-1}\left[\left(p+r+\frac{1}{2}\right)^2+\mu^2\right]\operatorname{sh}^{-p}\alpha_1\,\operatorname{sh}^{-p}\alpha_2\times$$

$$\times P^{-p-k}_{-\frac{1}{2}+i\mu}(\operatorname{ch}\alpha_1)\, P^{-p-k}_{-\frac{1}{2}+i\mu}(\operatorname{ch}\alpha_2). \tag{4}$$

In works (^{5},^{6}) an integral transform connected with the functions $P^{-p}_{-\frac{1}{2}+i\mu}(\operatorname{ch}\alpha)$ was studied. The group-theoretic meaning of this transform consists in expanding an arbitrary function $f(g)$, $g \in G$, $n = 2$, satisfying the functional equation $f(g) = f(\omega_1 g \omega_2)$, in an integral with respect to zonal spherical functions. Carrying out the corresponding considerations for any value of n, we obtain the following result.

Theorem 1. Let $\mathfrak{H}$ be the space of functions $f(\operatorname{ch}\alpha)$ for which the integral

$$\|f\|^2 = \int_0^{\infty} |f(\operatorname{ch}\alpha)|^2 \operatorname{sh}^{2p+1}\alpha\, d\alpha, \tag{5}$$

converges, and let $\mathfrak{R}$ be the space of functions $F(\mu)$ for which the integral

$$\|F\|^2 = \pi\int_0^{\infty} \frac{|F(\mu)|^2\, d\mu}{\mu\, \operatorname{sh}\pi\mu\, \Gamma\left(p+i\mu+\frac{1}{2}\right)\Gamma\left(p-i\mu+\frac{1}{2}\right)} \tag{6}$$

converges. Then the formulas

$$F(\mu) = \frac{\mu \operatorname{sh} \pi\mu}{\pi} \Gamma\left(p + i\mu + \frac{1}{2}\right) \Gamma\left(p - i\mu + \frac{1}{2}\right) \times$$

$$\times \int_0^\infty f(\operatorname{ch}\alpha)\, P^{-p}_{-\frac{1}{2}+i\mu}(\operatorname{ch}\alpha)\, \operatorname{sh}^{p+1}\alpha\, d\alpha; \tag{7}$$

$$f(\operatorname{ch}\alpha) = \operatorname{sh}^{-p}\alpha \int_0^\infty F(\mu)\, P^{-p}_{-\frac{1}{2}+i\mu}(\operatorname{ch}\alpha)\, d\mu \tag{8}$$

define mutually inverse mappings of the spaces $\mathfrak{H}$ and $\mathfrak{R}$ onto each other. Moreover, $\|f\|^2 = \|F\|^2$, i.e. the indicated mappings are isometric.

The inversion formulas (7) and (8) also lead to the relation

$$\frac{1}{\pi} \mu \operatorname{sh} \pi\mu\, \Gamma\left(p + i\mu + \frac{1}{2}\right) \Gamma\left(p - i\mu + \frac{1}{2}\right) \times$$

$$\times \int_0^\infty P^{-p}_{-\frac{1}{2}+i\lambda}(\operatorname{ch}\alpha)\, P^{-p}_{-\frac{1}{2}+i\mu}(\operatorname{ch}\alpha)\, \operatorname{sh}\alpha\, d\alpha = \delta(\mu - \lambda). \tag{9}$$

We shall now denote by $\mathfrak{A}$ the space of functions $f(\operatorname{ch}\alpha)$ for which the integral

$$\int_0^\infty |f(\operatorname{ch}\alpha)|\, \operatorname{sh}^{2p+1}\alpha\, d\alpha \tag{10}$$

converges.

We shall call the **convolution** of two functions $f_1(\operatorname{ch}\alpha)$ and $f_2(\operatorname{ch}\alpha)$ from $\mathfrak{A}$ the function

$$f(\operatorname{ch}\alpha) = f_1 * f_2(\operatorname{ch}\alpha) =$$

$$= \frac{\Gamma(p+1)}{\Gamma\left(p + \frac{1}{2}\right)\sqrt{\pi}} \int_0^\infty \int_0^\pi f_1(\operatorname{ch}\beta)\, f_2(\operatorname{ch}\alpha\ \operatorname{ch}\beta + \operatorname{sh}\alpha\ \operatorname{sh}\beta \cos\varphi) \sin^{2p}\varphi\, \operatorname{sh}^{2p+1}\beta\, d\varphi\, d\beta. \tag{11}$$

Then the function $f_1 * f_2(\operatorname{ch}\alpha)$ also belongs to $\mathfrak{A}$.

The generalized Fock-Mehler transform of the function $f(\operatorname{ch}\alpha)$ is the function

$$F(\mu) = \frac{2^p \Gamma(p+1) \pi F_1(\mu) F_2(\mu)}{\mu \operatorname{sh} \pi\mu\, \Gamma(p + i\mu + {}^1/_2) \Gamma(p - i\mu + {}^1/_2)}, \tag{12}$$

where $F_k(\mu)$ is the generalized Fock-Mehler transform of the function $f_k(\operatorname{ch}\alpha)$, $k = 1, 2$.

If, in formula (4), one takes α as the variable of integration and uses formula (8), then we obtain that the integral

$$\int_0^\infty \mu \operatorname{sh}\pi\mu\, \Gamma(p+k+{}^1\!/_2+i\mu)\Gamma(p+k+{}^1\!/_2-i\mu)\times$$

$$\times P^{-p-k}_{-{}^1\!/_2+i\mu}(\operatorname{ch}\alpha_1) P^{-p-k}_{-{}^1\!/_2+i\mu}(\operatorname{ch}\alpha_2) P^{-p}_{-{}^1\!/_2+i\mu}(\operatorname{ch}\alpha)\, d\mu \tag{13}$$

is equal to

$$\frac{2^{p-1}k!\Gamma(p)}{\Gamma(2p+k)} C_k^p\left(\frac{\operatorname{ch}\alpha - \operatorname{ch}\alpha_1 \operatorname{ch}\alpha_2}{\operatorname{sh}\alpha_1 \operatorname{sh}\alpha_2}\right) \frac{[\operatorname{ch}\alpha - \operatorname{ch}(\alpha_1-\alpha_2)]^{p-{}^1\!/_2}[\operatorname{ch}(\alpha_1+\alpha_2)-\operatorname{ch}\alpha]^{p-{}^1\!/_2}}{\operatorname{sh}^p\alpha_1 \operatorname{sh}^p\alpha_2 \operatorname{sh}^p\alpha}, \tag{14}$$

if $|\alpha_1 - \alpha_2| < \alpha < \alpha_1 + \alpha_2$, and is equal to zero if this condition is not satisfied.

Let us also note the relation

$$(\cos\varphi\ \operatorname{sh}\alpha + \operatorname{ch}\alpha)^{-p-{}^1\!/_2+i\mu} =$$

$$= \sum_{k=0}^\infty 2^p(p+k)\Gamma(p)\frac{\Gamma(-p+{}^1\!/_2+i\mu)}{\Gamma(-p-k+{}^1\!/_2+i\mu)} \operatorname{sh}^{-p}\alpha\, P^{-p-k}_{-{}^1\!/_2+i\mu}(\operatorname{ch}\alpha)\, C_k^p(\cos\varphi), \tag{15}$$

which follows from formula (2).

From the matrix elements (1), by passage to the limit, one can obtain the matrix elements of the Euclidean motion group, which are expressed in terms of Bessel functions. Therefore, from each formula derived in this paper one can obtain a corresponding formula for Bessel functions. We wish to note one formula related to Bessel functions, analogous to formula (13). The integral

$$\int_0^\infty J_{\nu+m}(ta)J_m(tb)J_\nu(tc)\, t\, dt, \qquad \nu > m-1, \tag{16}$$

where m is an integer and a, b, c are real numbers, is equal to zero if a, b, c are not sides of a triangle. If a, b, c are sides of a triangle, $a > b$, φ is the angle between a and b, ψ is the angle between a and c, and Δ is the area of the triangle, then this integral is equal to $\dfrac{1}{2\pi\Delta}\cos(\nu\psi - m\varphi)$.

Very interesting formulas are obtained when considering the group of affine transformations of $(n+1)$-dimensional space preserving the indefinite form $(x_1-y_1)^2+\ldots+(x_n-y_n)^2-(x_{n+1}-y_{n+1})^2$. Computing the zonal and associated spherical functions with respect to the (noncompact!) subgroup of homogeneous transformations, we obtain the following addition theorem:

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \mu\,\mathrm{sh}\,\mu\pi\,\Gamma(p+i\mu+{}^1/_2)\Gamma(p-i\mu+{}^1/_2)K_{i\mu}(x)K_{i\mu}(y)\times$$

$$\times P^{-p}_{-{}^1/_2+i\mu}(\mathrm{ch}\,\alpha)\,d\mu=\pi\left(\frac{xy}{z}\right)^{p+{}^1/_2}\mathrm{sh}^p\,\alpha\,K_{p+{}^1/_2}(z), \tag{17}$$

where $z^2=x^2+y^2+2xy\,\mathrm{ch}\,\alpha$.

Using the inversion formula (8), we find that

$$\frac{K_{i\mu}(x)K_{i\mu}(y)}{x^{p+1/2}y^{p+1/2}}=\sqrt{\frac{\pi}{2}}\int_0^\infty\frac{K_{p+1/2}(z)}{z^{p+1/2}}P^{-p}_{-1/2+i\mu}(\mathrm{ch}\,\alpha)\,\mathrm{sh}^{p+1}\,\alpha\,d\alpha. \tag{18}$$

With the aid of the Mehler inversion formula (7) we obtain from this that the integral

$$\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}I_{p+1/2}(Rz)K_{i\mu}(Rx)K_{i\mu}(Ry)R^{1/2-p}\,dR,\qquad C>0, \tag{19}$$

is equal to zero if $0\le x+y<z$, and is equal to

$$\sqrt{\frac{\pi}{2}}\,P^{-p}_{-1/2+i\mu}\left(\frac{z^2-x^2-y^2}{2xy}\right)\sqrt{\frac{(x+y+z)^p(z-x+y)^p(z+x-y)^p(z-x-y)^p}{(2z)^p xyz}}, \tag{20}$$

if $x+y>z$. Let us also note the formula

$$\frac{H^{(1)}_{i\mu}(x)H^{(1)}_{i\mu}(y)}{(xy)^{p+1/2}}=-i\sqrt{\frac{2}{\pi}}\,e^{\mu\pi}\int_0^\infty\frac{H^{(1)}_{-p-1/2}(z)}{z^{p+1/2}}P^{-p}_{-1/2+i\mu}(\mathrm{ch}\,\alpha)\,\mathrm{sh}^{p+1}\,\alpha\,d\alpha, \tag{21}$$

where $z^2=x^2+y^2+2xy\,\mathrm{ch}\,\alpha$, which likewise follows from considerations of representation theory. A number of very interesting formulas involving Legendre functions of the second kind arise in considering the group of homogeneous linear transformations preserving the form $x_1^2+x_2^2+\cdots+x_n^2-x_{n+1}^2+x_{n+2}^2$, and its subgroup acting in an $(n+1)$-dimensional space.

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named after V. V. Kuibyshev

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.