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Abstract

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MATHEMATICS

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ON THE REDUCTION OF UNBOUNDED NON-SELF-ADJOINT OPERATORS TO TRIANGULAR FORM

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1. The question of reducing a broad class of bounded non-self-adjoint operators to triangular form was solved by M. S. Livshits in papers ([1,2]). An analogous problem is considered here for unbounded operators.

Denote by G_A the set of vectors f belonging to the domain of definition D_A of the operator A and such that $(Af, g) = (f, Ag)$ for every $g \in D_A$. Let A_0 be the operator coinciding with A on G_A and defined only on G_A.

Definition. A closed operator A with everywhere dense domain of definition D_A will be called a **quasi-Hermitian operator of rank** r (in what follows, a K^r-operator) if A_0 is a Hermitian operator with defect index (r, r) $(r > 0)$ and $\dim D_A = r \pmod{G_A}$. We shall consider only those K^r-operators for which there exists at least one point λ regular together with its conjugate $(\operatorname{Im} \lambda \neq 0)$. Without loss of generality one may assume that $\lambda = i$. The class of K^r-operators contains all bounded non-self-adjoint operators, as well as many integral, differential, and other operators.

A K^r-operator A will be called a K^r_I-operator if $D_A = D_{A^*}$; a K^r_{II}-operator if $\overline{G}_A = H$, and a K^r_{III}-operator if $D_A \neq D_{A^*}$, $\overline{G}_A \neq H$. We note that the class of K^r_{II}-operators coincides with the class of quasi-self-adjoint extensions ([3]) of symmetric operators with equal defect numbers.

The largest invariant subspace ([4]) of the operator A, in which $A = A^*$, will be denoted by H_A and called the **supplementary subspace** of the operator A. The operator A_p, coinciding with A on the manifold $(H \ominus H_A) \cap D_A$, will be called the **simple part** of the operator A. The operator A is called **simple** if $A = A_p$.

2. Let $r = 1$. Consider a certain K^1-operator A and put

$$B = iR_{-i} - iR^*_{-i} - 2R^*_{-i}R_{-i}, \qquad R_{-i} = (A + iI)^{-1}.$$

The operator B may be represented in the form

$$Bf = J(f, g)g \quad (f \in H),$$

where $J = \pm 1$, $g \in \mathfrak{N}_{-i} = H \ominus \Delta_{A_0}(-i)$, $\Delta_{A_0}(-i) = (A_0 + iI)G_A$.

The function

$$\omega_A(\lambda) = 1 - (1 - \lambda i)((A^* - iI)(A^* - \lambda I)^{-1} g, g)J$$

will be called **characteristic** for the operator A.

Theorem 1. *Simple K^1-operators A_1 and A_2 are unitarily equivalent if and only if their characteristic functions coincide.*

In the upper half-plane the function $\omega_A(\lambda)$ satisfies the inequality

$$\omega_A(\lambda) J \overline{\omega_A(\lambda)} < J.$$

Using this property and the known representation of a bounded function regular in the unit disk, one can show,

that $\omega_A(\lambda)$ can be represented in the form

$$\omega_A(\lambda) = \prod_{k=1}^{N} \frac{L(k) - \lambda}{\overline{L(k)} - \lambda} \gamma(k) \exp\left[i \int_0^{\nu} \frac{1 + \lambda\alpha(t)}{\alpha(t) - \lambda}\, dt \right] \exp[i\lambda]\mu,$$

where $N \leq \infty$; $\nu < \infty$; $\mu \geq 0$;

$$L(k) = \alpha_k + \frac{i}{2}\beta_k^2 J; \qquad \gamma(k) = \frac{\overline{L(k)} - i}{L(k) - i}\, |L(k) - i|.$$

$\{\alpha_k\}$, $\{\beta_k\}$ are certain sequences of real numbers; $\alpha(t)$ is a nondecreasing numerical function.

Let $H_1 = l_2$, $H_2 = L_2(0, \nu)$, $H_3 = L_2(0, \mu)$.

Define in H_k $(k = 1, 2, 3)$ the operators A_k as follows:

$$A_1 f_1 = \left\{ f_1(k) L(k) + i \sum_{j=k+1}^{N} f_1(j)\beta(j) J \beta(k) \right\} \qquad (f \in H_1),$$

$$A_2 f_2 = \alpha(x) f_2(x) + i \int_x^{\nu} f_2(t)\gamma(t) J \overline{\gamma(x)}\, dt \qquad (f_2 \in H_2),$$

where $\gamma(x) = \alpha(x) + i$; moreover, we shall assume that $f_2 \in D_{A_2}$ if $A_2 f_2 \in H_2$, and

$$\int_0^{\nu} f_2(t)\gamma(t)\, dt < \infty.$$

Further,

$$A_3 f_3 = \frac{1}{i}\frac{df_3(x)}{dx},$$

where $f_3 \in D_{A_3}$ if:

1) $f_3(x)$ is absolutely continuous on $[0,\mu]$; 2) $f_3(x), f_3'(x) \in H_3$; 3) $f_3(\mu) = 0$.

Consider now in $H = H_1 \oplus H_2 \oplus H_3$ the manifold D_A, consisting of vectors f of the form

$$P_1 f = f_1 + \left[\frac{\sqrt{2}}{2} i \int_0^\nu f_2(t)\gamma(t)\,dt - \frac{1}{2} f_3(0) e^{J\nu}\right] g, \tag{1}$$

$$P_2 f = f_2(x) - \frac{1}{2} f_3(0,\, -\nu + x)\frac{\alpha(x) - i}{\alpha(x) + i}, \qquad P_3 f = f_3(x),$$

where $f_k \in D_{A_k}$ $(k = 1, 2, 3)$;

$$g = \left\{ \frac{\beta_k}{L(k) + i} \prod_{j=k+1}^{N} \frac{\overline{L(j)} + i}{L(j) + i}\gamma(j) \right\};$$

P_k is the projection operator onto H_k $(k = 1, 2, 3)$.

On the manifold D_A define the operator A as follows: if the vector f is determined by the equalities (1), then

$$P_1 A f = A_1 f_1 - i\left[\frac{\sqrt{2}}{2} i \int_0^\nu f_2(t)\gamma(t)\,dt - \frac{1}{2} f_3(0) e^{J\nu}\right] g,$$

$$P_2 A f = A_2 f_2 + \frac{i}{2} J f_3(0) e^{J(\nu + x)} \frac{\alpha(x) - i}{\alpha(x) + i}, \qquad P_3 A f = A_3 f_3.$$

We shall call the operator A a **triangular model** of the K^1-operator A.

The following assertions hold:

1) The operator A is a K^1-operator.

2) The characteristic function of the operator A coincides with the characteristic function of the operator A.

3) The nonreal spectrum of the operator A consists of the set of numbers $L(k)$, which are eigenvalues of the operator A; the real spectrum coincides with the set of values assumed by the function $\alpha(x)$.

From Theorem 1 and assertion 2) it follows:

Theorem 2. *For every K^1-operator A there exists an isometric operator V, mapping $H \ominus H_A$ onto $\mathbf{H} \ominus \mathbf{H}_A$ one-to-one and such that the simple part A_p of the operator A is carried under this mapping into the simple part $\mathbf{A}_p$ of its triangular model.*

We note that the operators A_1, A_2, A_3, by means of which the operator $\mathbf{A}$ is defined, are K^1-operators. Here the operators A_1 and A_2 outwardly resemble the discrete and continuous parts of the triangular model of bounded non-self-adjoint operators (1). If in equality (1) $\mu = 0$ and

$$\sum_{k=1}^{N} \beta_k + \int_0^{\nu} |\alpha(x)|^2 \, dx < \infty,$$

then the operator A is a K_1^1-operator and can be transformed to the same form as that of the triangular model for bounded operators. Conversely, if A is a K_1^1-operator, then in equality (1) $\mu = 0$ and

$$\sum_{k=1}^{N} \beta_k^2 + \int_0^{\nu} |\alpha(x)|^2 \, dx < \infty.$$

Consequently, the triangular model for K_1^1-operators coincides (in outward form) with the model for bounded operators.

An operator A, defined in $H = H_1 \oplus H_2$, will be called a **coupling of the operators** A_1 and A_2, acting in H_1 and H_2, respectively, if $A|_{D_A \cap H_1} = A_1$, $[A^*|_{D_{A^*} \cap H_2}]^* = A_2$. The operator considered earlier is a coupling of the operators A_1, A_2, A_3. In the general case, the triangular model of a K^r-operator A ($r < \infty$) is a coupling of the operators A_1, A_2, A_3, A_4, which are defined in certain (specified) spaces of vector-functions as follows:

$$A_1 f_1 = \left\{ f_1(k)L(k) + i \sum_{j=k+1}^{N} f_1(j)\sigma(j)J\sigma^*(k) \right\}, \qquad A_2 f_2 = B_2^{-1} Q_2 J B_2^{-1} f_2,$$

where $J f_2 = f_2(x) J$,

$$Q_2 f_2 = \frac{1}{i} \frac{d f_2(x)}{dx} \quad (0 \le x \le \mu),$$

and $f_2 \in D_{Q_2}$ if:

1) $f_2(x)$ is absolutely continuous on $[0, \mu]$;

2) $f_2(x), f_2'(x) \in H_2$;

3) $f_2(\mu) = 0$.

The operator B_2 is defined by the equality $B_2 f_2 = f_2(x)P(x)$; moreover, the operator A_2 is considered in the subspace $\widetilde{H}_2 = H_2 \ominus H_0$, where H_0 is the largest subspace annihilating the operator B_2.

The operator A_3 is defined by the equality

$$A_3 f_3 = \alpha(x) f_3(x) + i \int_x^\nu f_3(t)\gamma(t)\omega(t)\omega^{-1}(x)J\gamma^*(x)\,dt.$$

The operator A_4 is defined in the same way as the operator A_2.

The elements by means of which the operators A_k $(k = 1, 2, 3, 4)$ are defined are taken from the multiplicative representation of the characteristic matrix-function of the K^r-operator A for which the model is constructed.

For lack of space we are unable to give the definition of the characteristic matrix-function and to give a complete description of the triangular model for an arbitrary K^r-operator $(r < \infty)$. We note, however, that in the general case the definition of the characteristic matrix-function differs little from the case $r = 1$. The multiplicative representation mentioned may be obtained easily by using the main theorem of V. P. Potapov (5). For every $r < \infty$ the same assertions hold as in the case $r = 1$.

It is interesting to note that in the triangular model for K^1-operators, in contrast to the triangular model for bounded operators, there appears–

a substantially new element–the differentiation operator. In the general case, two essentially new operators appear in the triangular model (A_2 and A_4), reminiscent of the differentiation operator. The presence of two, and not one, such operators for $r > 2$ is connected with the nonpermutability of the matrices entering into the multiplicative representation of the characteristic matrix-function for K^r-operators.

In conclusion I take the opportunity to express my deep gratitude to my supervisor, Prof. M. S. Livshits.

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Note: Figure translations are in progress. See original paper for figures.