Electrical Engineering

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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

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A Probability-Theoretic Study of a Two-Stage Telephone System with Losses Operating in the Free-Search Mode

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1. In automatic telephony, two-stage connection is widely used; one of its variants is considered in the present paper by the methods of the theory of homogeneous Markov processes with continuous time and a finite number of possible states (1–3). The results obtained are a multidimensional analogue of the well-known Erlang formulas (3–5).

2. Consider a two-stage scheme consisting of k switches of the first stage and m switches of the second stage. Suppose that switch i of the first stage has n_i inputs $(i = 1, 2, \dots, k)$ and m outputs, and each of the switches of the second stage has k inputs and l outputs $(k, m, n_1, n_2, \dots, n_k \geqslant 1)$ (see Fig. 1). Let a Poisson stream of calls with parameter λ_i arrive at the inputs of switch i of the first stage $(i = 1, 2, \dots, k)$, the streams arriving at different switches being mutually independent. Each of the arriving calls may be directed along any of the free connecting paths to one of the switches of the second stage, where this call may occupy any of the free outputs (in telephone terminology, the free-search mode). The order in which free lines are seized may be arbitrary—random and equiprobable, or sequential. A call arriving at switch i of the first stage is lost if all inputs are occupied (when $n_i < m$) or all connecting paths (when $n_i > m$) of this switch leading to the second stage are occupied, and also if all outputs of those switches of the second stage with which it can be connected at the given moment are occupied. Suppose that the holding time of each line of input switch i has a negative-exponential distribution with mean value μ_i $(i = 1, 2, \dots, k)$, and that the holding time of one line is independent of the holding times of the other lines, as well as of the incoming calls.

Fig. 1

3. Denote by s_i the number of occupied lines of switch i of the first stage, and

by $\mathbf{P}\{(s_1, s_2, \dots, s_k); t\}$ the probability that at time t $(t > 0)$ the system is in the state $(s_1, s_2, \dots, s_k)$. We shall also regard the initial probabilities of the system at $t = 0$ as given. It follows from the preceding point that the random process thus defined will be a homogeneous Markov transient process $((^4), Ch.6)$.

First consider the case $l = 1$. For brevity, putting $\min(n_i, m) = a_i$, we write the following obvious conditions:

$$0 \leqslant s_i \leqslant a_i, \qquad i = 1, 2, \dots, k, \tag{1}$$

$$0 \leqslant \sum_1^k s_i \leqslant \min\left(m, \sum_1^k a_i\right) = m. \tag{2}$$

In the case $m \geqslant \sum_1^k a_i$, condition (2) follows from (1) (see item 4).

These conditions mean that the point $(s_1, s_2, \dots, s_k)$ is located inside or on the surface of the part of the k-dimensional parallelepiped (1) cut off by the hyperplane $\sum_1^k s_i = m$. The set of points with integer coordinates determined by conditions (1) and (2) will be denoted by Ω. In what follows we shall assume that $\mathbf{P}\{(s_1, s_2, \dots, s_k); t\} \equiv 0$ $(t \geqslant 0)$ at all points of the k-dimensional space not belonging to Ω (condition A).

The transition probabilities of the process over a small time interval Δt will be as follows:

$$\mathbf{P}\{(s_1, \dots, s_i, \dots, s_k), \Delta t, (s_1, \dots, s_i + 1, \dots, s_k)\} =$$

$$= \begin{cases} \lambda_i \Delta t + o(\Delta t), & \text{if } (s_1, \dots, s_i + 1, \dots, s_k) \in \Omega, \\ 0, & \text{if } (s_1, \dots, s_i + 1, \dots, s_k) \notin \Omega, \end{cases} \qquad i = 1, 2, \dots, k$$

(one call arrived at switch i of the first stage);

$$\mathbf{P}\{(s_1, \dots, s_i, \dots, s_k), \Delta t, (s_1, \dots, s_i - 1, \dots, s_k)\} = \frac{s_i \Delta t}{\mu_i} + o(\Delta t),$$

$$(s_1, \dots, s_i, \dots, s_k) \in \Omega, \qquad i = 1, 2, \dots, k$$

(one conversation passing through switch i of the first stage ended);

$$\mathbf{P}\{(s_1, \dots, s_i, \dots, s_k), \Delta t, (s_1, \dots, s_i, \dots, s_k)\} =$$

$$= 1 - \sum_1^k {}' \lambda_i \Delta t - \sum_1^k \frac{s_i \Delta t}{\mu_i} + o(\Delta t), \qquad (s_1, \dots, s_i, \dots, s_k) \in \Omega$$

(no call arrived and no conversation ended).

The remaining transition probabilities have a higher order of smallness.

Let us note that in the sum $\sum_1^k {}' \lambda_i$, the terms λ_i for which $s_i = a_i$ are equal to zero, and that $\sum_1^k {}' \lambda_i = 0$ if $\sum_1^k s_i = m$ (condition B).

Using now the Markov property of the process and the transition probabilities over time Δt, we obtain:

$$\mathbf{P}\{(s_1, \dots, s_i, \dots, s_k); t + \Delta t\} = \sum_{i=1}^k \mathbf{P}\{(s_1, \dots, s_i - 1, \dots, s_k); t\} \lambda_i \Delta t +$$

$$+ \sum_{i=1}^k \mathbf{P}\{(s_1, \dots, s_i + 1, \dots, s_k); t\} \frac{(s_i + 1)\Delta t}{\mu_i} +$$

$$+ \left(1 - \sum_{i=1}^k {}' \lambda_i \Delta t - \sum_{i=1}^k \frac{s_i \Delta t}{\mu_i}\right) \mathbf{P}\{(s_1, \dots, s_i, \dots, s_k); t\} + o(\Delta t). \tag{3}$$

Passing in (3) to the limit as $\Delta t \to 0$, we obtain the following system of differential equations:

$$\frac{d\mathbf{P}\{(s_1, \dots, s_i, \dots, s_k); t\}}{dt} = \sum_{i=1}^k \lambda_i \mathbf{P}\{(s_1, \dots, s_i - 1, \dots, s_k); t\} +$$

$$+ \sum_{i=1}^k \frac{s_i + 1}{\mu_i} \mathbf{P}\{(s_1, \dots, s_i + 1, \dots, s_k); t\} -$$

$$- \left(\sum_{i=1}^k {}' \lambda_i + \sum_{i=1}^k \frac{s_i}{\mu_i}\right) \mathbf{P}\{(s_1, \dots, s_i, \dots, s_k); t\} \tag{4}$$

with boundary conditions A and B, an initial probability distribution $\mathbf{P}\{(s_1, s_2, \dots, s_k); 0\}$ in Ω, and the normalizing condition

$$\sum_\Omega \mathbf{P}\{(s_1, s_2, \dots, s_k); t\} = 1 \quad (t \geq 0).$$

The number of equations in system (4) is equal to the number of unknowns.

From the transience of the Markov process under consideration $((^2), \S 4; (4), Ch.6)$ there follows the existence and uniqueness of the limiting probabilities

$p(s_1, s_2, \dots, s_k)$, which do not depend on the initial probability distribution and satisfy the system of linear homogeneous equations

$$\sum_{i=1}^{k} \lambda_i p(s_1, \dots, s_i - 1, \dots, s_k) + \sum_{i=1}^{k} \frac{s_i + 1}{\mu_i} p(s_1, \dots, s_i + 1, \dots, s_k) -$$

$$- \left(\sum_{i=1}^{k}{}' \lambda_i + \sum_{i=1}^{k} \frac{s_i}{\mu_i} \right) p(s_1, \dots, s_i, \dots, s_k) = 0 \tag{5}$$

with boundary conditions A and B and the normalizing condition

$$\sum_{\Omega} p(s_1, s_2, \dots, s_k) = 1. \tag{6}$$

By direct substitution it is easy to verify that

$$p(s_1, s_2, \dots, s_k) = p(0, 0, \dots, 0) \prod_{i=1}^{k} \frac{(\lambda_i \mu_i)^{s_i}}{s_i!}, \qquad (s_1, s_2, \dots, s_k) \in \Omega, \tag{7}$$

is a solution of the system of equations (5) with boundary conditions A and B. The value $p(0, 0, \dots, 0)$ is determined from condition (6):

$$\frac{1}{p(0, 0, \dots, 0)} = \sum_{\Omega} \prod_{i=1}^{k} \frac{(\lambda_i \mu_i)^{s_i}}{s_i!}. \tag{8}$$

For $k = 1$, formulas (7) pass into Erlang' s formulas $(0 \le s_1 \le a_1)$.

4. Let us now consider the case $l \ge k$. In this case losses are possible only because of the absence of free connecting paths between switches of the first and second stages, so that the probability of loss for a call arriving at switch i of the first stage depends only on the number of busy lines of this switch and does not depend on the number of busy lines of the remaining switches of the first stage. Therefore

$$\mathbf{P}\{(s_1, s_2, \dots, s_k); t\} = \prod_{i=1}^{k} \mathbf{P}(s_i; t), \qquad 0 \le s_i \le a_i, \quad i = 1, 2, \dots, k,$$

   where each of the k mutually independent probabilities $\mathbf{P}(s_i; t)$ has, as $t \to \infty$, an Erlang limiting distribution, and the limiting probabilities of the composite process are determined by formulas (7) and (8), with the difference that the region Ω is determined only by condition (1). This result also holds for $l = 1$, if $m \ge \sum_1^k a_i$. The case $1 < l < k$ is not considered in this article.

5. Introduce the random variable $\xi_t(s_1, s_2, \dots, s_k)$—the time spent by the system in state $(s_1, s_2, \dots, s_k)$ over the time interval t. Let

$M_\gamma \xi_t(s_1, \dots, s_k)$ be the mathematical expectation of the random variable $\xi_t(s_1, \dots, s_k)$ under the condition that at the initial moment of time the system was at the point $\gamma \in \Omega$. From the transience of the Markov process under consideration it follows $((^2), \S 4)$ that, independently of the initial state,

$$\lim_{t\to\infty} \frac{M_\gamma\, \xi_t(s_1, \dots, s_k)}{t} = p(s_1, \dots, s_k), \qquad (s_1, \dots, s_k) \in \Omega. \tag{9}$$

The ergodic theorem (9) clarifies the physical meaning of the final probabilities $p(s_1, \dots, s_k)$.

6. We shall now derive formulas that make it possible to obtain the probability distribution of the total number of busy lines. First consider the case

$$l = 1, \quad m < \sum_1^k a_i.$$

Denote

$$s_1 + s_2 + \dots + s_k = r, \qquad p(s_1 + s_2 + \dots + s_k = r) = p_r, \qquad p(0, 0, \dots, 0) = p_0.$$

Obviously,

$$p_r = p_0 \sum_{\substack{s_1+\dots+s_k=r \\ 0\le s_i\le a_i}} \prod_{i=1}^k \frac{(\lambda_i\mu_i)^{s_i}}{s_i!}, \qquad 0 \le r \le m. \tag{10}$$

If

$$\min_{i=1,2,\dots,k} n_i = a < m,$$

then for $0 \le r \le a$ formula (10) simplifies to:

$$p_r = \frac{\Lambda^r/r!}{\sum\limits_{r=0}^{a} \frac{\Lambda^r}{r!} + \sum\limits_{r=a+1}^{m} \sum\limits_{\substack{s_1+\dots+s_k=r \\ 0\le s_i\le a_i}} \prod\limits_{i=1}^k \frac{(\lambda_i\mu_i)^{s_i}}{s_i!}}, \tag{11}$$

where

$$\Lambda = \sum_1^k \lambda_i\mu_i.$$

It follows from formula (11) that when $a \ge m$, formulas (10) and (11) become Erlang' s formulas for a full-availability group of m lines, to which a Poisson stream of calls arrives with intensity

$$\sum_1^k \lambda_i.$$

The parameter of the Erlang formulas is Λ:

$$p_r = \frac{\Lambda^r}{r!} \Big/ \sum_{r=0}^{m} \frac{\Lambda^r}{r!}, \qquad 0 \le r \le m.$$

In the case

$$l = 1, \quad m \ge \sum_{1}^{k} a_i,$$

and also in the case $l \ge k$, analogous formulas hold for p_r

$$\left(0 \le r \le \sum_{1}^{k} a_i\right),$$

with the difference that the region Ω is determined only by condition (1).

7. We shall say that at time τ the system is in state α if at that moment exactly α of the k switches of the first stage are busy ($\alpha = 0, 1, \dots, k$). Accordingly, the set Ω decomposes into a sum of disjoint subsets T_α, which admit a simple geometric interpretation.

Denote

$$w_\alpha = \sum_{T_\alpha} p(s_1, \dots, s_k), \qquad \sum_{\alpha=0}^{k} w_\alpha = 1.$$

Here w_k is the probability of complete occupancy of the system, and w_0 is the probability of its failure-free operation. It follows from the ergodic theorem that, as $t \to \infty$ and independently of the initial state, w_α is the limit of the mean relative time during which the system remains in state α. The quantity

$$\sum_{\alpha=1}^{k} \alpha w_\alpha$$

may be regarded as an indicator of the average occupancy of the system.

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Note: Figure translations are in progress. See original paper for figures.