THE GENERAL CASE OF STABILITY OF CHARACTERISTIC EXPONENTS AND THE EXISTENCE OF LEADING COORDINATES

Consider systems of differential equations

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Abstract

Full Text

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THE GENERAL CASE OF STABILITY OF CHARACTERISTIC EXPONENTS AND THE EXISTENCE OF LEADING COORDINATES

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Consider systems of differential equations

$$y' = P(t)y; \tag{1}$$

$$x' = P(t)x + \varphi(t, x). \tag{2}$$

Here x is a vector; $P(t)$ is an $n \times n$ matrix with elements bounded for $0 \leq t < \infty$; $\varphi(t, x)$ is a vector of nonlinear perturbations, $\varphi(t, 0) \equiv 0$, satisfying the Lipschitz condition in x with "constant" $g(t)$. The nonlinearity is considered small if $g(t) = g_1(t) + g_2(t)$, where

$$\int_0^\infty g_1 \, d\xi < \infty,$$

and $g_2(t)$ is equal to or less than a sufficiently small δ, or tends to zero. The matrix $P(t)$ is henceforth assumed to be purely diagonal with real elements

$$p_1(t),\ p_2(t), \ldots, p_n(t). \tag{3}$$

This is no restriction, since the general case is always reduced to the indicated one by a Perron transformation and a so-called σ-transformation, as a result of which $g(t)$ acquires only a finite and easily computable multiplier.

The problem is posed of finding conditions that should be imposed on the functions (3) in order to ensure the stability of characteristic exponents, i.e. the closeness of the exponents of systems (1) and (2) under arbitrary perturbations with sufficiently small δ. Works of a number of authors are devoted to this problem in its various aspects ([1−5]).

In ([1,2]) it is assumed that (3) are constants; the result is stability of the exponents. The assumptions in ([3]) are the separatedness of the functions (3):

$$p_{i+1}(t) - p_i(t) \geq a > 0, \qquad i = 1, 2, \dots, n-1, \tag{4}$$

the linearity of $\varphi(t, x)$, and $g(t) \to 0$; the result is the coincidence of the exponents of (1) and (2) and the existence of leading coordinates along each axis, i.e. the existence, for any i, of such a solution $(x_1, x_2, \dots, x_n)$ that $x_j/x_i \to 0$. In ([4]) the separatedness condition is weakened to $p_{i+1} - p_i > 0$ and

$$\int_0^\infty (p_{i+1} - p_i)\, d\xi = \infty,$$

but this weakening is compensated by the additional requirement $g/(p_{i+1} - p_i) \to 0$ instead of $g \to 0$; the results are similar to ([3]). The most general assumptions are those in ([5]), where instead of (4) only integral separatedness is required:

$$\int_\tau^t (p_{i+1} - p_i)\, d\xi \geq a(t - \tau) - B \quad \text{for all } \tau < t,\ i = 1, 2, \dots, n-1, \tag{5}$$

with positive a and B; the result is the stability of the exponents. However, in this work an additional restriction is essentially used—the regularity of system (1)—without which the proof does not go through. Thus, here the results of ([1,2]) are generalized, but not those of ([3,4]).

Nevertheless, reducing the separation requirement to the level of (5) is of particular interest. The point is that, in connection with the study of the central exponent ([6]), in the questions under consideration the predominance of the role not of the functions (3) themselves, but of their Steklov averages,

$$p_i^H(t) = H^{-1} \int_t^{t+H} p_i(\xi)\, d\xi$$

with large H, has become clear. It is easy to verify that condition (5) is equivalent to the condition of ordinary separation for p_i^H: $p_{i+1}^H(t) - p_i^H(t) \geq a_1 > 0$ for $H \geq H_1$; the latter, moreover, is "almost necessary" for the stability of exponents: if it is violated for all large H on a set of positive relative measure (we call a set $A \subset [0, \infty)$ such if $\lim_{T\to\infty} T^{-1} \operatorname{mes}(A \cap [0, T]) > 0$), then the exponents are certainly unstable.

Therefore it seemed useful to determine the decisive significance of the condition of integral separation (5) for the stability of exponents, without any additional assumptions on the regularity of (1), linearity of $\varphi(t, x)$, the tendency of $g(t)$ to zero, etc.

The following theorem serves this purpose.

Theorem. Suppose that (5) is fulfilled. Whatever $\varepsilon > 0$, $\gamma > 0$ may be, there exists such a $\delta > 0$ that for any perturbations with $g_2(t) < \delta$ the solutions of system (2) possess the following properties:

1°. The norm of every solution has the form

$$|x(t)| = \exp\left[c(t) + \int_0^t (p_i + \varepsilon_x)\, d\xi\right] \tag{6}$$

with one of $i = 1, 2, \dots, n$, where $|c(t)|$ is bounded, $|\varepsilon_x(t)| < \varepsilon$; in particular, the exponent of such a solution lies in the interval $(\underline{p}_i - \varepsilon, \overline{p}_i + \varepsilon)$,

$$\overline{p} = \overline{\lim}\, t^{-1} \int_0^t p\, d\xi.$$

2°. The set $E^k(t_0)$ of initial points (for every $t_0 \geq 0$) of solutions with exponents $< \overline{p}_k + \varepsilon$ is homeomorphic to the k-dimensional plane $E^k = \{x_1, x_2, \dots, x_k\}$; the manifolds $E^k(t_0)$ are nested: $E^k(t_0) \subset E^{k+1}(t_0)$.

3°. All solutions from 2° lie in the "cone of width γ about the plane E^k," i.e., for them

$$(x_{k+1}^2 + \dots + x_n^2)/(x_1^2 + \dots + x_k^2) < \gamma^2.$$

This holds, starting from $t = 0$, if

$$\int_0^\infty g_1\, d\xi$$

is sufficiently small, and otherwise—starting from some t_1, common to all the solutions under consideration.

4°. Every solution of the form (6) is, moreover, immersed in the "cone of width γ about the axis x_i," and in this sense its i-th coordinate is leading; however, the immersion occurs starting from large t, depending on the solution.

5°. For every $t_0 \geq 0$ there exists a single homeomorphism between the initial points of solutions of (1) and (2), under which the exponents of corresponding solutions coincide to within $\pm\varepsilon$.

6°. If instead of $g_2 \leq \delta$ we have $g_2 \to 0$, then the corrections $\pm\varepsilon$ are removed everywhere, and the width of the cones γ may be regarded as arbitrarily small for large t.

In the proof the following lemma is used.

Lemma on the Riccati equation. Let there be given the equation (with one unknown y)

$$y' = p(t)y - G(t)(1 + y^2), \tag{7}$$

where $|p(t)| \le M$,

$$\int_\tau^t p\,d\xi \ge a(t-\tau) - B, \qquad G(t) = \delta_0 + g_1(t).$$

If

$$\int_0^\infty g_1\,d\xi = \delta_1 < +\infty$$

and δ_0, δ_1 are sufficiently small, then for arbitrarily large T there exists a solution of (7) satisfying the conditions:

I. $\delta_0/M < y(t) < M/\delta_0 + 1$ for all t.

II. $y(t) < q$ for $0 \le t \le T,\ 0 < q < 1$.

III. $y(t) > (1-q)/q$ for t equal to or greater than some $T_1 > T$.

Let us briefly outline the proof of the theorem. It is carried out by induction on n as follows. All solutions of system (2) are divided into two classes: the first consists of those for which

$$x_n^2(t) < x_1^2(t) + \cdots + x_{n-1}^2(t)$$

for all t; the second consists of the remaining ones; it is proved that the classes are nonempty. It is easy to establish that the projection of every solution of the first class onto the plane E^{n-1} satisfies a system of the form (2) of dimension $n-1$, so that for this projection, by the induction hypothesis, assertions 1°–6° are fulfilled. Hence, by the method of integral equations of Grobman ([2]), refining the arguments of ([5]), one succeeds in deriving that the whole solutions of the first class also satisfy 1°–6°. If a solution belongs to the second class, then for some $T \ge 0$ we have

$$x_n^2(T) \ge x_1^2(T) + \cdots + x_{n-1}^2(T). \tag{8}$$

We construct the cone

$$x_n^2 \ge y_1^2(t)x_1^2 + \cdots + y_{n-1}^2(t)x_{n-1}^2,$$

where $y_i(t)$ is a solution of the Riccati equation

$$y_i' = [p_n(t) - p_i(t)]y_i - G(t)(1 + y_i^2),$$

chosen, according to the lemma, for the indicated T. From relations (8) and condition II of the lemma it is seen that at the moment $t = T$ the solution under study lies inside the cone. The main point of the proof consists in the fact that, for sufficiently small $\delta < \delta_0$, the vectors of the field (2) on the boundary of the cone turn out everywhere to be directed inward (taking into account the variability of the cone in time t), and therefore the chosen solution does not leave the cone. Then from condition III of the lemma it is clear that for $t \geq T_1$ it lies in the "cone of width $q/(1-q)$ about the axis x_n," and, if q is small, then the width does not exceed the prescribed γ. The remaining assertions of the theorem are obtained from this without any particular difficulty.

The theorem generalizes the results mentioned above from ([1,2,5]) (in the case of distinct exponents) and ([3]). It may also be extended to the case of coincidence of some exponents, with a complication of the formulations.

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Note: Figure translations are in progress. See original paper for figures.