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Abstract

Full Text

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ON THE ZEROS OF SOLUTIONS OF THE EQUATION $y^{(n)} + p(x)y = 0$

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We shall consider the equation

$$y^{(n)} + p(x)y = 0 \quad (1)$$

either on the segment $[a, b]$, or on the half-line $[a, +\infty)$. The coefficient $p(x)$ will be assumed continuous. The case $n = 3, 4$ was considered by the author in the paper ([1]).

We shall say that, for equation (1), **condition A** is satisfied if every solution of it either has an infinite number of zeros, or tends monotonically to zero ([2]). Obviously, condition A is meaningful on $[a, +\infty)$. It is easy to establish that, in the case $p(x) > 0$ and even n, the fulfillment of condition A is equivalent to the fact that every solution has infinitely many zeros. In the case $p(x) > 0$ and odd n, however, it is established that there always exists a solution tending monotonically to zero.

Theorem 1. *If $p(x) \geq q(x) > 0$ and for the equation*

$$y^{(n)} + q(x)y = 0 \quad (2)$$

condition A is satisfied, then it is also satisfied for equation (1).

Introduce the set $E_{x_0,s(x)}$. We shall say that $x \in E_{x_0,s(x)}$ if there exists a nonnegative solution on $[x_0, x]$ of the equation $y^{(n)} + s(x)y = 0$ such that $y(x_0) = y(x) = 0$. The least upper bound of the set $E_{x_0,s(x)}$, if it is finite, will be denoted by $\tau_{x_0,s(x)}$. It is not difficult to prove that the set $E_{x_0,s(x)}$ is closed, and, in the case when condition A is satisfied for the equation $y^{(n)} + s(x)y = 0$, this set is bounded; hence $\tau_{x_0,s(x)} \in E_{x_0,s(x)}$. In particular, $\tau_{x_0,q(x)} \in E_{x_0,q(x)}$.

Let us consider the equation

$$y^{(n)} + p_1(x)y = 0, \quad (3)$$

where $p_1(x) \equiv p(x)$ on $[a, \tau_{x_0,q(x)} + 1]$, and $p_1(x) = \max[q(x); p(\tau_{x_0,q(x)} + 1)]$ on $[\tau_{x_0,q(x)} + 1, \infty)$.

Since $p_1(x) \geq p(\tau_{x_0,q(x)} + 1) > 0$, condition A is satisfied for equation (3) ([2]); $\tau_{x_0,p_1(x)} \in E_{x_0,p_1(x)}$, and therefore there exists on $[x_0, \tau_{x_0,p_1(x)}]$ a nonnegative solution of equation (3) vanishing at the endpoints of this segment. It can be shown that among such solutions there is one, $\bar{y}(x)$, which on $[x_0, \tau_{x_0,p_1(x)}]$ has exactly n zeros, each counted according to its multiplicity. Denote its zeros by α_i ($\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$). It is further proved that the solution of equation (2) having zeros at the points α_i ($i = 1, 2, \ldots, n-1$) on $[x_0, \tau_{x_0,p_1(x)}]$ does not change sign; hence $\tau_{x_0,q(x)} > \tau_{x_0,p_1(x)}$. It follows from this that every solution of equation (3) having a zero at the point x_0 changes sign at some point of the interval $(x_0, \tau_{x_0,q(x)})$, and since on this interval $p_1(x) \equiv p(x)$, the solution of equation (1) that vanishes at the point x_0 also vanishes on $(x_0, \tau_{x_0,q(x)})$. Since x_0 may be taken as an arbitrary point, every solution of the equation

(1), having at least one zero, has an infinite number of zeros. Hence it follows easily that condition A is satisfied for equation (1).

Indeed, in the case of even n we can extend $p(x)$ to $x < a$ so that every solution of equation (1) vanishes to the left of a, whence it will follow that every solution has an infinite number of zeros. For this it is enough to set $p(x) = p(a)$ for $x < a$. If n is odd, then, as we have already noted, there is a solution $y_1(x)$ tending monotonically to zero, and if some solution $y_2(x)$ has no zeros, then one can show that it tends monotonically to zero. If this is not so, then, by virtue of (1) and the fact that $p(x) > 0$, the solution $y_2(x)$ is monotone; therefore the solution

$$y(x) = y_1(a)y_2(x) - y_2(a)y_1(x)$$

cannot oscillate, but it vanishes at the point a, and hence oscillates by what was proved above; this is a contradiction.

We shall call equation (1) **nonoscillatory** if each of its solutions has no more than $n-1$ zeros.

Theorem 2. *If there exist functions $p_1(x)$ and $p_2(x)$ such that $p_2(x) \leq p(x) \leq p_1(x)$, and the equations $y^{(n)} + p_1(x)y = 0$, $y^{(n)} + p_2(x)y = 0$ are nonoscillatory, then equation (1) is nonoscillatory.*

The proof is based on the fact that if $y^{(n)} + s(x)y = 0$ is nonoscillatory, then

$$y^{(n)} + s(x)y = \frac{d}{dx}s_n(x)\cdots\frac{d}{dx}s_2(x)\frac{d}{dx}s_1(x)y,$$

where $s_i(x) > 0$ ([3]). Hence

$$y^{(n)} + p(x)y \equiv y^{(n)} + p_1(x)y + [p(x) - p_1(x)]y \equiv$$

$$\equiv \frac{d}{dx}r_n(x)\cdots\frac{d}{dx}r_2(x)\frac{d}{dx}r_1(x)y + (p - p_1)y = 0, \tag{4}$$

and analogously:

$$y^{(n)} + p(x)y = \frac{d}{dx}r_n\cdots\frac{d}{dx}r_2\frac{d}{dx}r_1y + (p - p_2)y = 0. \tag{5}$$

From equalities (4) and (5) it is concluded that a solution of equation (1) having a zero of multiplicity k at the point a cannot have a zero of multiplicity $n-k$ for $x > a$. It is proved that the latter is equivalent to the fact that $W_{1\,2,3,\ldots,l}(x)$ –the Wronskian of solutions $y_1, y_2, \ldots, y_l$ such that

$$y_i^j(x_0) = 0, \qquad j \neq n-i;$$
$$y_i^j(x_0) = 1, \qquad j = n-i,$$

has no zeros to the right of a for any $l \leq n$. We set $W_1 = y_1(x)$. As shown in ([3]), the existence of such a chain of Wronskians is sufficient for nonoscillation.

Let us note that if, in the hypotheses of Theorem 2, one requires only the fulfillment of a single inequality $p(x) \leq p_1(x)$, it can be proved that equation (1) has a fundamental system consisting of solutions having no more than $n-1$ zeros, although the equation may already fail to be nonoscillatory.

Let us apply Theorems 1 and 2, taking as comparison equations the Euler equation

$$y^{(n)} + \frac{k}{x^n} y = 0.$$

Let λ_k be the maxima of the function

$$f(\alpha) = -\prod_{i=0}^{n-1} (\alpha - i)$$

on the interval $[0, n-1]$, and let μ_k be its minima. Further, let $\bar{\lambda}$ be the largest of the numbers λ_k, and λ the smallest; and, analogously, let $\bar{\mu}$ be the largest of the μ_k, and μ the smallest.

From Theorems 1 and 2 it follows that if $p(x) \geqslant \dfrac{\bar{\lambda}+\varepsilon}{x^n}$, then condition A is satisfied for equation (1); if $\dfrac{\bar{\mu}}{x^n} \leqslant p(x) \leqslant \dfrac{\lambda}{x^n}$, equation (1) is nonoscillatory. Finally, one can prove that if $p(x) \leqslant \dfrac{\mu - \varepsilon}{x^n}$, then there exists a fundamental system of solutions of equation (1) such that $3/2 + 1/2(-1)^n$ of the solutions belonging to it have a finite number of zeros, and the remaining ones have an infinite number.

Let us note that if $n < 5$, then $\lambda = \bar{\lambda}$, $\mu = \bar{\mu}$.

In the case $n = 3, 4$ and $p(x) > 0$, it was proved in [2] that between two consecutive zeros of one solution of equation (1) there lie at most two or, respectively, four zeros of another solution. For $n > 4$ no analogous theorem can be obtained.

It can be shown that, whatever $m > 0$ may be, there exists $p(x) > 0$ such that between two consecutive zeros of one solution there lie more than m zeros of another. For this it is necessary to put

$$p(x) = \frac{\lambda'}{x^n}$$

on $[a, k]$, where λ' is any number such that $\lambda < \lambda' < \bar{\lambda}$; $p(x) = p(k)$ on $[k, \infty)$, and k is chosen sufficiently large. Equation (1) in this case can be solved on $[a, k]$. Its solutions are the functions x^{α_i}, where α_i are the roots of the equation $\lambda' = f(\alpha)$, among which there are at least two real roots α_1, α_2 and two complex roots α_3, α_4. The solution

$$e^{\alpha_2 a} e^{\alpha_1 x} - e^{\alpha_1 a} e^{\alpha_2 x}$$

has a zero at the point a; on $(a, k]$ it has no zeros and is monotone; whereas the solution $e^{\alpha_4 x}$, for sufficiently large k, has more than m zeros on (a, k). The solution of equation (1) which on $[a, k]$ coincides with

$$e^{\alpha_2 a} e^{\alpha_1 x} - e^{\alpha_1 a} e^{\alpha_2 x}$$

has at least one zero to the right of k, since on $[k, \infty)$ it coincides with a certain solution of the equation

$$y^{(n)} + \frac{\lambda'}{k^n} y = 0,$$

and if it did not oscillate, it would tend monotonically to zero; but since it vanishes at the point a, its n-th derivative changes sign, which cannot occur by virtue of (1) and the fact that $p(x) > 0$.

In conclusion let us note the connection between the asymptotic growth of solutions and the number of their zeros. We assume $p(x) > 0$. Denote by $y_1(x)$ the solution having at $\{x_1$ a zero of multiplicity $n-1$, and by $y_2(x)$ the solution having at the same point a zero of multiplicity $n-2$, and let $W_{12}(x)$ be the Wronskian of these solutions. It is proved that

$$W_{12}(x) > c(x - x_1)^{2n-4},$$

where $c > 0$. Further, one can show that if

$$r = \sqrt{y_1^2 + y_2^2},$$

then

$$y_1 = r \cos \int_{x_1}^{x} \frac{W_{12}(x)}{r^2}\, dx; \qquad y_2 = r \sin \int_{x_1}^{x} \frac{W_{12}(x)}{r^2}\, dx.$$

If $R(x)$ is such that $r(x) \leqslant R(x)$, then there exists a sequence x_k such that

$$N(x_1, x_k) \geqslant c \int^{x_k} \frac{(x - x_1)^{2n-4}}{R^2}\, dx,$$

where $N(x_1, x_k)$ is the number of zeros of y_i, $i = 1, 2$, on $[x_1, x_k]$.

In particular, if all solutions are bounded, the relation

$$N(x_1, x) \leqslant o(x^{2n-3})$$

cannot hold. Hence, if equation (1) is nonoscillatory, all its solutions cannot be bounded.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.