



Soviet-era science, translated into English

MATHEMATICS

A. E. GEL' MAN

1958

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-195801.07659>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

MATHEMATICS

A. E. GEL' MAN

THE METHOD OF A SMALL PARAMETER FOR OPERATOR EQUATIONS

(Presented by Academician V. I. Smirnov on 11 VII 1958)

Lemma. Let

$$F(\lambda, x) = a + \lambda \sum_{i,j=0}^{\infty} a_{ij} x^i \lambda^j,$$

where $a, a_{ij} \geq 0$; ρ, R are positive radii of convergence with respect to λ and x . Then, if $a < R$, there exists a unique root of the equation

$$x = F(\lambda, x), \quad (1)$$

representable in the form of the series

$$x(\lambda) = \sum_{k=0}^{\infty} x_k \lambda^k.$$

Moreover $x_k \geq 0$, and the radius of convergence of the series $x(\lambda)$ (if $F(\lambda, 0) \neq 0$)* is determined by the formula

$$\Lambda = \sup_{0 < x < R} \lambda,$$

where λ and x are connected by equation (1).

If, moreover, the function $F(\lambda, x)$ is nonlinear with respect to x , then the series $x(\lambda)$ also converges at the point $\lambda = \Lambda$ (in this case Λ is finite)**.

Let Y be a space of type B , and let Y_λ be the linear system of all formally constructed power series of the form

$$y(\lambda) = \sum_{k=0}^{\infty} y_k \lambda^k,$$

where $y_k \in Y$. We shall write

$$y(\lambda) = \sum_{k=0}^{\infty} y_k \lambda^k \ll \sum_{k=0}^{\infty} x_k \lambda^k,$$

if

$$\|y_k\| \leq x_k.$$

In the case where the series $\sum_{k=0}^{\infty} x_k \lambda^k$ has a nonzero radius of convergence, we shall denote its sum by $x(\lambda)$,

$$x(\lambda) = \sum_{k=0}^{\infty} x_k \lambda^k$$

and write

$$y(\lambda) \ll x(\lambda).$$

* If $F(\lambda, 0) = 0$, then the trivial case $x(\lambda) = 0$ occurs (i.e. $x_k = 0$).

** This fact is extremely important for estimating the remainder term of the series $x(\lambda)$.

Theorem 1. Let the operator Ω_λ satisfy the following conditions:

1) Ω_λ maps Y_λ into itself, and

$$\Omega_\lambda[y(\lambda)] = \Omega_\lambda(y_0 + y_1 \lambda + y_2 \lambda^2 + \dots) = \Omega_0(0) + \sum_{k=1}^{\infty} \lambda^k \omega_k(y_0, y_1, \dots, y_{k-1}),$$

where ω_k is an operator mapping the set of k -dimensional vectors with components from Y into Y (i.e., if $y_0, y_1, \dots, y_{k-1} \in Y$, then $\omega_k(y_0, y_1, \dots, y_{k-1}) \in Y$).

2) There exists a multiple power series

$$\bar{\Omega}(\lambda, x) = a + \lambda \sum_{i,j=0}^{\infty} a_{ij} x^i \lambda^j$$

with positive radii of convergence such that, for $y(\lambda) \ll x(\lambda)$, $x(0) < R$, where R is the radius of convergence with respect to x of the series $\bar{\Omega}(\lambda, x)$, the relation

$$\Omega_\lambda[y(\lambda)] \ll \bar{\Omega}[\lambda, x(\lambda)]$$

holds.

3) $\|\Omega_0(0)\| < R$.

Then:

a) The equation

$$y = \Omega_\lambda(y)$$

has a unique solution $y(\lambda)$ belonging to Y_λ :

$$y(\lambda) = y_0 + y_1\lambda + y_2\lambda^2 + \dots, \quad (2)$$

where $y_k \in Y$.*

- b) The series (2) converges for $|\lambda| < \Lambda^{**}$, where $\Lambda = \sup_{0 < x < R} \lambda$, and λ and x are connected by the equation

$$x = \bar{\Omega}(\lambda, x). \quad (3)$$

- c) The relation

$$y(\lambda) \preceq x(\lambda)$$

holds, where $x(\lambda)$ is the unique root, analytic with respect to λ , of equation (3).

Assertion a) of this theorem is obvious; assertions b) and c) are proved by constructing a majorant series and using the lemma.

Theorem 2. Let the operators L and ω_λ satisfy the following conditions:

- 1) L and ω_λ map some linear subsystem \tilde{Y}_λ of the system Y_λ into Y_λ .***
- 2) The operator L has an inverse L^{-1} , mapping Y_λ into \tilde{Y}_λ . Moreover, there exists a power series $\bar{L}^{-1}(x)$ with positive radius of convergence such that, for $y(\lambda) \preceq x(\lambda)$ and $x(0) < R_1$, where R_1 is the radius of convergence of $\bar{L}^{-1}(x)$, the relation

$$L^{-1}[y(\lambda)] \preceq \bar{L}^{-1}[x(\lambda)]$$

holds.

- 3) The operator $\Omega_\lambda = \omega_\lambda L^{-1}$ satisfies all the conditions of Theorem 1.

* Assertion a) follows only from condition (1).

** If the function $\bar{\Omega}(\lambda, x)$ is essentially nonlinear with respect to x , the series also converges for $\lambda = \Lambda$ (in this case Λ is finite).

*** The theorem remains valid also in the case when ω_λ maps \tilde{Y}_λ into Y_λ .

Then:

- a) The equation $L(y) = \omega_\lambda(y)$ has a unique solution $y(\lambda) \in \tilde{Y}_\lambda$:

$$y(\lambda) = y_0 + y_1\lambda + y_2\lambda^2 + \dots \quad (4)$$

b) The series (4) converges for $|\lambda| < \Lambda$, where

$$\Lambda = \sup_{0 < x < \min(R, R_1)} \lambda,$$

λ and x being connected by the equation

$$x = \overline{\Omega}(\lambda, x). \quad (5)$$

c) The relation

$$y(\lambda) \ll \overline{L^{-1}}[x(\lambda)]$$

holds, where $x(\lambda)$ is the unique root of equation (5) analytic with respect to λ .

Theorem 2 follows easily from Theorem 1. Quite essential for applications is the fact that the requirements of Theorem 1 are imposed precisely on the operator $\omega_\lambda L^{-1}$ (and not on the operator $L^{-1}\omega_\lambda$).

Remark 1. The estimates obtained by us for the radii of convergence are attained within the class of equations under consideration (this is clear from the lemma).

Remark 2. We know only a few works ⁽¹⁻⁵⁾ devoted to estimating the radius of convergence of series obtained by using the small-parameter method in the theory of ordinary differential equations. Works ⁽¹⁻⁴⁾ are devoted to the study of special cases of Theorem 2; the estimates obtained in these works either coincide with the estimates of Theorem 2* (for example, in ⁽⁴⁾), or are cruder than them ⁽¹⁻³⁾. The result of work ⁽⁵⁾ can also be obtained as a special case of Theorem 2 (and, moreover, improved), if Theorem 2 is generalized by introducing in the proper way the notion of a generalized inverse operator.

Remark 3. Theorems 1 and 2 are proved without applying the method of successive approximations. The quantitative part of these theorems can also be proved with the aid of this method (although less naturally).

It seems to us, however, that the results obtained cannot be placed within the general classical scheme of L. V. Kantorovich ⁽⁶⁾.

Remark 4. In very common cases, when $\Omega_\lambda(y)$ is linear with respect to λ , i.e. when

$$\Omega_\lambda(y) = \Omega_0(0) + \lambda\omega(y),$$

Λ is found especially simply:

$$\Lambda = \sup_{0 < x < \min(R, R_1)} \frac{x - \|\Omega_0(0)\|}{\omega(x)}.$$

Remark 5. The results of the present work may be applied to the study of the problem of stability of solutions of operator equations, to nonlinear problems of mathematical physics, etc.

Leningrad Electrotechnical Institute
named after V. I. Ulyanov (Lenin)

Received
25 VI 1958

REFERENCES

1. A. Poincaré, *Acta Math.*, **13** (1890).
2. A. A. Kruming, *Ukr. Mat. Zh.*, **5**, No. 4 (1953).
3. D. S. Lewis, *Duke Math. J.*, **22**, 1 (1955).
4. Yu. A. Ryabov, *DAN*, **118**, No. 4 (1958).
5. A. E. Gelman, *DAN*, **118**, No. 6 (1958).
6. L. V. Kantorovich, B. Z. Vulikh, V. G. Pinsker, *Functional Analysis in Semi-Ordered Spaces*, Moscow–Leningrad, 1950.

* With a proper choice of the norm and of majorizing functions.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.