

FRACTIONAL POWERS OF DIFFERENTIAL OPERATORS AND EMBEDDING THEOREMS

1958

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Abstract

Full Text

MATHEMATICS

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FRACTIONAL POWERS OF DIFFERENTIAL OPERATORS AND EMBEDDING THEOREMS

(Presented by Academician S. L. Sobolev, 5 VI 1958)

1. Let G be a bounded domain of n -dimensional space ($n \geq 2$), star-shaped with respect to all points of some ball contained in it. In the Hilbert space $L_2(G)$ we shall consider a self-adjoint positive-definite operator A , generated by some differential operator of even order and by a system of homogeneous boundary conditions.

We shall say that the operator A is **strongly invertible** if

$$\|A^{-1}f\|_{W_2^l} \leq C\|f\|_{L_2}, \quad (f \in L_2), \quad (1)$$

where $\| \cdot \|_{W_2^l}$ is the norm in the Sobolev space W_2^l .

As shown by Caccioppoli ⁽¹⁾ and Ladyzhenskaya ⁽²⁾ for a second-order operator, and by Gusev ⁽³⁾ for operators of higher order, elliptic operators are strongly invertible for a broad class of domains and boundary conditions.

From Sobolev's embedding theorems ⁽⁴⁾ and inequality (1) one can conclude that the operator A^{-1} maps the space L_2 into the space C , or into some space L_q , where q is determined by the numbers n and l . We shall be interested in the question of into which spaces the operators $A^{-\gamma}$, $0 < \gamma < 1$, map L_2 .

Theorem 1. *Let A be a strongly invertible operator, $0 < \gamma < 1$, and $r = \gamma l - n/2$. The following cases are possible:*

- a) *The number r is positive and nonintegral. Then the operator $A^{-\gamma}$ is a completely continuous operator acting from the space L_2 into the space $C_{m,\nu}$ of functions having partial derivatives $m = [r]$, satisfying the Hölder condition with exponent $\nu < r - [r]$.*
- b) *r is a positive integer. Then the operator $A^{-\gamma}$ is completely continuous and acts from L_2 into the space $C_{m,\nu}$, where $m = r - 1$ and $\nu < 1$.*
- c) *$r \leq 0$. Then the operator $A^{-\gamma}$ is a completely continuous operator acting*

from L_2 into L_q , where

$$\frac{1}{q} > -\frac{r}{n} = \frac{1}{2} - \frac{\gamma l}{n}.$$

Let us note that assertion c) of the theorem can be obtained from assertion a) by means of M. A. Krasnosel'skii's theorem ⁽⁵⁾ on fractional powers of operators.*

The role played by estimates of Green's functions of differential operators is well known. However, in a number of applications it is in fact

* Fractional negative powers of the Laplace operator were studied by V. A. Il'in ⁽⁶⁾, from whose results it follows that $A^{-\gamma}$ for $\gamma > n/4$ acts from L_2 into C and is bounded.

what is needed is not these estimates themselves, but information about the functional space in which the operator $D^m A^{-1}$ acts, where D^m is a certain partial derivative of order m . We shall consider this question for the operator $D^m A^{-\gamma}$.

Theorem 2. *Let A be a strongly invertible operator, and let m be a positive integer satisfying the inequality $\gamma l - \frac{n}{2} < m < \gamma l$; then the operator $D^m A^{-\gamma}$ is a completely continuous operator acting from L_2 into L_q , where*

$$\frac{1}{q} > \frac{1}{2} - \frac{\gamma l - m}{n}. \quad (2)$$

Combining the assertions of Theorem 1 with Theorem 2 makes it possible to conclude that, under the hypotheses of the latter theorem, the operator $A^{-\gamma}$ acts into the space W_q^m .

Let M be a point of the domain \bar{G} . Consider the operator

$$D_h^m f(P) \equiv \frac{1}{|M - P|^h} D^m f(P) \quad (h \geq 0).$$

With the aid of Hölder's inequality, Theorem 2 yields a conclusion about where the operator $D_h^m A^{-\gamma}$ acts. However, for $\gamma = 1$ one can obtain a more precise result.

Theorem 2'. *Let A be a strongly invertible operator and let the nonnegative integer m satisfy the inequalities $l - \frac{n}{p} < m < l$. Then, for every h such that $m + h < l$, the operator $D_h^m A^{-1}$ is a bounded operator acting from L_p into L_q , where*

$$\frac{1}{q} = \frac{1}{p} - \frac{l - m - h}{n}.$$

If $m = l - \frac{n}{p}$, then the preceding assertion is true for $q > p$, $h < \frac{n}{q}$.

Following (7), we shall call the **order α of the operator D_h^m relative to the operator A** the lower exact bound of the numbers γ for which the operator $D_h^m A^{-\gamma}$ is bounded in L_2 .

Theorem 3. For $0 \leq m < l$ and $0 \leq h < \min\{l - m, \frac{n}{2}\}$, the operator D_h^m is an operator of fractional order not exceeding $\frac{m+h}{l}$ relative to the operator A , and, moreover, for $\frac{m+h}{l} < \gamma < 1$,

$$\left\| \frac{1}{|M-P|^h} D^m A^{-\gamma} \varphi \right\|_{L_2} \leq K \|\varphi\|_{L_2},$$

where K does not depend on $M \in G$.

Using Theorem 3 and the results of (7), one can study solutions of elliptic and parabolic equations in which the coefficients of the lower (spatial) derivatives have point singularities. In this case, the sum of the order of the derivative and the order of the singularity of the coefficient standing with it must be less than the order of the principal terms.

2. We present some facts that are used in the proofs of Theorems 1, 2, 3. At the basis of these proofs lies a small refinement of a theorem established in (7) by P. E. Sobolevskii and one of the authors.

Let E be a space embedded everywhere densely in L_2 , and suppose $\|\varphi\|_{L_2} \leq C\|\varphi\|_E$. Let B be an operator defined on $D(A) \subset E$ and admitting a closure \bar{B} as an operator from L_2 into E . In order that, for all $\gamma > \alpha$, the inequality

$$\|\bar{B}A^{-\gamma}\varphi\|_E \leq K(\gamma, E)\|\varphi\|_{L_2},$$

it is necessary and sufficient that, for sufficiently small δ and $\gamma > \alpha$, the inequality

$$\|Bf\|_E \leq \frac{K_1(\gamma, E)}{\delta^\gamma} \|f\|_{L_2} + \delta^{1-\gamma} \|Af\|_{L_2} \quad (f \in D(A)). \quad (3)$$

hold.

Part c) of Theorem 1 follows from the assertion just given, if B is taken to be the identity operator. Inequality (3) is then obtained from known inequalities of this type for generalized partial derivatives of various orders. These inequalities were obtained by various authors under one or another set of assumptions concerning the boundary of the domain G . They are also valid under our assumptions.*

Part a) of Theorem 1 follows from Theorem 3 with the aid of a theorem of embedding type established by Nirenberg ((⁸), Lemma 1). To obtain part b) of Theorem 1 from Theorem 3 it was necessary to supplement Nirenberg's result; this was achieved with the aid of a theorem on operators of potential type established by one of the authors of the present paper.

For the proof of Theorems 2' and 3 it was necessary to establish inequalities of type (3) no longer for partial-derivative operators, but for the operators D_h^m . We shall state them in a more general form than is needed for Theorems 2' and 3.

Theorem 4. Let $\varphi \in W_p^l$ and $s = l - \frac{n}{p}$. Then, for all sufficiently small $\delta > 0$ and $l > m > s$, the inequality

$$\|D_h^m \varphi\|_{L_q} \leq \frac{K}{\delta^\chi} \|\varphi\|_{L_p} + \delta^{1-\chi} \|\varphi\|_{L_p^l}, \quad (4)$$

holds, where

$$\chi = \frac{n}{l} \left(\frac{1}{p} - \frac{1}{q} \right) + \frac{m+h}{l}, \quad (5)$$

and K does not depend on δ , φ , or the point $M \in \overline{G}$ with respect to which the operator D_h^m is constructed. Here $0 \leq h \leq l - m$, $\frac{1}{p} > \frac{1}{q} \geq \frac{1}{p} - \frac{l-m-h}{n}$.

Theorem 4'. Let $\varphi \in W_p^l$ and let $s = l - \frac{n}{p} > 0$. Then, for all sufficiently small $\delta > 0$, the inequality

$$\sup_{P, Q \in G} \frac{|D^m \varphi(P) - D^m \varphi(Q)|}{|P - Q|^\nu} \leq \frac{K}{\delta^\chi} \|\varphi\|_{L_p} + \delta^{1-\chi} \|\varphi\|_{L_p^l},$$

holds, where

$$\chi = \frac{m+h}{l} + \frac{n}{pl}$$

and K does not depend on δ or φ .

Here m is the greatest integer smaller than s , and $\nu = s - [s]$ if s is not an integer, while $\nu < 1$ if s is an integer.

Theorems 4 and 4' are obtained with the aid of certain theorems on operators of potential type and S. L. Sobolev's representation (⁴) of functions from W_p^l . In doing so one has to use certain geometric properties of domains that are star-shaped with respect to a ball.

The spaces W_p^l consist of functions which themselves belong to L_p and are mapped into L_p by all partial-derivative operators D^l of order l . Similarly, one may consider the space of functions $\varphi \in L_p$ for which $D_k^l \varphi \in L_p$ for given l and k . Theorems 4 and 4' generalize to such spaces.

* In the most general form these inequalities were apparently obtained in an unpublished work of V. P. Il' in, with which he kindly acquainted the authors.

3. The complete continuity of the operators discussed in Theorems 1 and 2 is established with the aid of inequalities analogous to the well-known Friedrichs inequality for the Laplace operator ⁽⁹⁾.

Theorem 5. *Under the conditions of Theorem 4 the inequality holds*

$$\|D_h^m \varphi\|_{L_q} \leq \frac{K}{\delta^{\varkappa + \frac{2q}{p'}}} \sum_{i=1}^N \left| \int \varphi(P) \omega_i(P) dP \right| + \delta^{1+\varkappa} \|\varphi\|_{L_p} \quad \left(\frac{1}{p} + \frac{1}{p'} = 1 \right),$$

where \varkappa is determined by formula (5); K does not depend on φ , δ , or M ; N does not depend on φ ; the $\omega_i(P)$ are functions bounded by a single constant independent of φ .

Under the conditions of Theorem 4' there is a theorem analogous to Theorem 5.

4. As follows from ⁽¹⁰⁾, Theorems 1, 2, 3 may be useful in the study of elliptic and parabolic equations with nonlinearities, as well as ⁽¹¹⁾ in the question of expansions in series in eigenfunctions of differential operators.

For these problems it would be important to know into which spaces the operators $D_h^m A^{-\gamma}$ act not only from L_2 , but also from L_p for $p > 2$. To obtain such theorems by the methods of the present work, a further strengthening of the result of ⁽⁷⁾ on operators of fractional order is required.

Received
4 VI 1958

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