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Abstract

Full Text

Mathematics

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On the Spectra of Dynamical Systems Generated by Stationary Gaussian Processes

(Presented by Academician A. N. Kolmogorov on 21 I 1958)

Let $x(t, \omega) = \int e^{i\lambda t} \Phi(d\lambda, \omega)$ be a real stationary (in the narrow sense) process, $F(d\lambda) = M(|\Phi(d\lambda, \omega)|^2)$ its spectral measure; the shift $S_\tau x(t, \omega) = x(t + \tau, \omega)$ preserves measure in the space of trajectories of the process and therefore defines a dynamical system, to which there corresponds a group of unitary operators U^τ in the Hilbert space H of real functionals of the trajectories of the process.

The spectrum of U^τ is described by the maximal spectral type ρ and the multiplicity function $\nu(\lambda)$. (We shall everywhere denote a measure and its spectral type by the same letter.) If $\nu(\lambda) \equiv 1$, then the spectrum is called *simple*; we shall say that the spectrum has a *simple component* if there exists a set A such that $\rho(A) > 0$, $\nu(\lambda) = 1$ for $\lambda \in A$.

In the case of a discrete measure $F(d\lambda)$, the structure of the spectrum of U^τ has been studied completely. The aim of the present note is a partial solution of the problem proposed by A. N. Kolmogorov at the seminar on dynamical systems—to investigate the case of a Gaussian process with continuous $F(d\lambda)^*$, having support without commensurabilities, and also the more general case of a continuous measure whose support has a measurable basis (see § 3).

We shall rely on the following theorem of Ito, proved by him for processes with continuous measure $F(d\lambda)$.

Theorem. $H = \bigoplus_0^\infty H_k$, where H_k is an invariant subspace with respect

to U^τ , isomorphic to the space \tilde{L}_k^2 of complex functions $f(\lambda_1, \dots, \lambda_k)$, square-integrable with respect to the measure $F(d\lambda_1) \dots F(d\lambda_k)$, symmetric with respect to permutations of $\lambda_1, \dots, \lambda_k$, and such that**

$$f(\lambda_1, \dots, \lambda_k) = \overline{f(-\lambda_1, \dots, -\lambda_k)}.$$

The image of U^τ in \tilde{L}_k^2 is V_k^τ , where

$$V_k^\tau f(\lambda_1, \dots, \lambda_k) = e^{i(\lambda_1 + \dots + \lambda_k)\tau} f(\lambda_1, \dots, \lambda_k).$$

With the aid of this theorem one can construct the following examples.

§ 1. **A process with simple continuous spectrum***. Let X be a perfect set on the real line, symmetric

* Everywhere below we shall write “process,” meaning by this a stationary Gaussian process.

** This condition is caused by the reality of the process. The formulation and proof of Ito’s theorem apply to the complex case, but can be easily modified for a real process.

*** In the examples known so far the spectrum is countably multiple (i.e. $\nu(\lambda) = \aleph_0$ (see (3)).

relative to zero, and such that among its positive elements there are no rational relations (2); $F_0(d\lambda)$ is a continuous finite measure, symmetric with respect to zero, whose support is X (the distribution function of such a measure can be constructed after the manner of the “Cantor staircase”). Then the process with spectral measure $F(d\lambda) = F_0(d\lambda)$ has a simple spectrum; its maximal spectral type will be, as was proved in (3),

$$\rho = \sum_1^{\infty} F_0^i,$$

where $F_0^i = F_0 * \dots * F_0$ is the i -fold composition of the measure F_0 ; the cyclic vector in H can be expanded in a series of Itô stochastic integrals of all orders.

The proof of this follows from the following facts: first, the measures F_0^i are mutually singular (see Lemma 1); second, every function from \bar{L}_k^2 can be replaced by an equivalent function of the sum $\lambda_1 + \dots + \lambda_k$. This latter fact can be done because the set

$$\underbrace{X \otimes \dots \otimes X}_k$$

in k -dimensional Euclidean space is projected onto the axis $\lambda_1 = \lambda_2 = \dots = \lambda_k$ one-to-one up to permutations of coordinates, except for a set of F_0^k -measure zero on this axis.

p. 2. **A process with a nonhomogeneous spectrum.** Put now

$$F(d\lambda) = F_0(d\lambda) + F_0^2(d\lambda).$$

The corresponding Gaussian process will have the same maximal type ρ and

$$\nu(\lambda) = \sum_{k=\lfloor \frac{n+1}{2} \rfloor}^n \frac{n!}{2^{n-k}(n-k)!(2k-n)!} \quad \text{for } \lambda \in X^n \setminus \sum_1^{n-1} X^k,$$

where X^i denotes the totality of points $\lambda = \lambda_1 + \dots + \lambda_i$, $\lambda_r \in X$. An analogous, but more cumbersome, formula can be obtained for

$$F = \sum_{i=1}^{\infty} F_0^{p_i}.$$

Examples are easily constructed: 1) with $\nu(\lambda)$ taking only the values 1 and \aleph_0 ; 2) a process for which $\nu(\lambda) \equiv 1$, but rational relations exist between the points of the support of the measure $F(d\lambda)$.

p. 3. We shall say that a **measure F has a measurable basis** if there exists a set X , among whose positive elements there are no rational relations, measurable together with all X^i , $i \geq 1$, and such that

$$F(\Delta) = F\left(\Delta \cap \bigcup_1^{\infty} X^k\right)$$

for every measurable Δ . The set X is called a **basis of the measure F** .

By the **order of the measure F relative to the basis X** we shall mean the largest k such that $F(X^i - h) = 0$ for $i < k$ and all h ($X^i - h$ denotes the set X^i shifted by h). If the measure is discrete, then we put its order equal to 0.

Theorem 1. *If the spectral measure of the process $x(t, \omega)$ is continuous and has finite order, then the spectrum of the dynamical system determined by this process contains a simple component.*

In the proof of Theorem 1 the following lemmas are used; they are also of independent interest.

Lemma 1. *Let G_1, \dots, G_s be finite measures with orders k_1, \dots, k_s relative to the basis X . Then the order of $G = G_1 * \dots * G_s$ is equal to $k_1 + \dots + k_s$.*

Lemma 2. *If the spectral measure F of the process $x(t, \omega)$ is subordinated**

* Independently of the author, this result was obtained by Yu. Rozanov.

$\sum_2^{\infty} F^i$, then $\nu(\lambda) \equiv \aleph_0$; if the measure F contains a component singular with respect to $\sum_2^{\infty} F^i$, then the spectrum contains a simple component.

To prove Theorem 1, denote by k the order of F and note that, by Lemma 1, the order of $\sum_2^{\infty} F^i$ is $2k > k$, since $k \geq 1$ by the continuity of F . Consequently,

F contains a component singular with respect to $\sum_2^\infty F^i$, and we may apply Lemma 2.

Corollary. A continuous measure with a measurable basis always has finite order; hence it follows that there is a simple component in the spectrum of the corresponding process.

Theorem 2. *The multiplicity function $\nu(\lambda)$ of the spectrum of the dynamical system of a process with continuous $F(d\lambda)$ is either identically equal to 1 or increases without bound.*

Indeed, if $\nu(\lambda) > 1$ on a set of positive measure

$$\rho = \sum_1^\infty F^i,$$

then one can find two elements φ_1 and φ_2 , lying in H_{k_1} and H_{k_2} , respectively (where, possibly, $k_1 = k_2$), such that the subspaces spanned by $U^\tau \varphi_1$ and $U^\tau \varphi_2$ are orthogonal, while U^τ has in them one and the same spectral type ρ_0 ; by Itô's theorem, φ_1 and φ_2 correspond to functions $f_1(\lambda_1, \dots, \lambda_{k_1})$ and $f_2(\lambda_1, \dots, \lambda_{k_2})$ in $\tilde{L}_{k_1}^2$ and $\tilde{L}_{k_2}^2$. Then all

$$f_{p,q} = f_1(\lambda_1^{(1)}, \dots, \lambda_{k_1}^{(1)}) \cdots f_1(\lambda_1^{(p)}, \dots, \lambda_{k_1}^{(p)}) f_2(\lambda_1^{(p+1)}, \dots, \lambda_{k_2}^{(p+1)}) \cdots f_2(\lambda_1^{(p+q)}, \dots, \lambda_{k_2}^{(p+q)})$$

for different p, q such that $p + q = s$, $p \geq 0$, $q \geq 0$, generate orthogonal invariant subspaces with spectral type ρ_0^s , and therefore $\nu(\lambda) \geq s + 1$ on some set of positive measure.

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CITED LITERATURE

- ¹ K. Itô, *Japan. J. Math.*, **22**, 63 (1952).
- ² J. Neymann, *Math. Ann.*, **99**, No. 1, 134 (1928).
- ³ S. V. Fomin, *Ukr. matem. zhurn.*, **2**, 2, 25 (1950).

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