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1958

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Abstract

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INVESTIGATION OF A FUNCTIONAL ANALOGUE OF ONE NONLINEAR INTEGRAL EQUATION OF LICHTENSTEIN

(Presented by Academician A. N. Kolmogorov, 10 VII 1957)

In Lichtenstein' s monograph (^{1}) an integral operator L was considered that is a broad generalization of the Fredholm operator. It is expressed by an infinite sum of multiple integrals of increasing multiplicity and is generated by variation of an analogously expressed functional F. Lichtenstein proved the existence of one nontrivial eigen-element and eigenvalue for the operator L in Hilbert space.

Generalizing the result of L. A. Lyusternik (^{2}) to the case of the unit ball of a regular Banach space, in our papers (^{3,4}), in particular, we showed that the functional F, under general conditions, is weakly continuous and satisfies the conditions for the existence of a critical element. Thereby, under more general assumptions, the existence of an eigen-element and eigenvalue of the Lichtenstein operator was proved.

Moreover, if the functional F generating the integral operator L is even (i.e., F is represented by integrals of even multiplicity) and $F > 0$, then L has an infinite set of positive eigenvalues tending to zero.

In the present note we investigate a class of infinite systems of nonlinear functional equations representing an analogue of the Lichtenstein integral equation in the real space l_2.

Let S_1 be the unit closed ball of the space l_2 with elements

$$x = (x_{\alpha_i}), \qquad \sum_{\alpha_i=0}^{\infty} x_{\alpha_i}^2 < \infty.$$

Consider a functional $F(x)$, defined for elements $x \subset S_1$, of the form

$$F(x) = \frac{1}{n+1} \sum_{\alpha_0,\dots,\alpha_n=1}^{\infty} a_{\alpha_0\cdots\alpha_n}\, \varphi(x_{\alpha_0}) \cdots \varphi(x_{\alpha_n}), \tag{1}$$

where $a_{\alpha_0\cdots\alpha_n}$ are given coefficients symmetric in the aggregate of indices $\alpha_0, \dots, \alpha_n$; φ is a twice continuously differentiable function on the segment $[-1, +1]$; n is a natural number.

We shall assume that

$$\sum_{\alpha_0,\dots,\alpha_n=1}^{\infty} a^2_{\alpha_0\cdots\alpha_n} < \infty, \qquad \sum_{\alpha_k=1}^{\infty} \varphi^2(x_{\alpha_k}) < \infty, \qquad \sup_{x\subset S_1} \sum_{\alpha_k=1}^{\infty} \varphi^2(x_{\alpha_k}) = M^2. \tag{2}$$

Then the $(n+1)$-fold series (1) converges uniformly in S_1.

Indeed, successively using Hölder' s inequality, from (1) we obtain

$$F(x) \le \frac{1}{n+1} \prod_{j=0}^{n} \left\{ \sum_{\alpha_j=1}^{\infty} \varphi^2(x_{\alpha_j}) \right\}^{1/2} \left\{ \sum_{\alpha_0,\dots,\alpha_n=1}^{\infty} a^2_{\alpha_0\cdots\alpha_n} \right\}^{1/2}.$$

Hence, by virtue of (2), we have

$$F(x) \le \frac{M^{n+1}}{n+1} \left\{ \sum_{\alpha_0,\dots,\alpha_n=1}^{\infty} a^2_{\alpha_0,\dots,\alpha_n} \right\}^{1/2} < \infty.$$

Lemma 1. *The functional $F(x)$ is weakly continuous in S_1.*

Let $\{x^{(k)}\} \in S_1$ be an arbitrary sequence weakly converging to the weak limit x^*, $\|x^*\| \le 1$. Denote $x^{(k)} = (x^{(k)}_\alpha)$ and $x^* = (x^*_\alpha)$; then $x^{(k)}_\alpha \to x^*_\alpha$ as $k \to \infty$. Starting from the transformation

$$\begin{aligned} F(x^{(k)}) - F(x^*) &= \frac{1}{n+1} \sum_{\alpha_0,\dots,\alpha_n=1}^{\infty} \left[\varphi(x^{(k)}_{\alpha_0}) \dots \varphi(x^{(k)}_{\alpha_n}) - \varphi(x^*_{\alpha_0}) \cdots \varphi(x^*_{\alpha_n}) \right] \\ &= \frac{1}{n+1} \sum_{\alpha_0,\dots,\alpha_n=1}^{\infty} a_{\alpha_0\cdots\alpha_n} \Big\{ [\varphi(x^{(k)}_{\alpha_0}) - \varphi(x^*_{\alpha_0})] \varphi(x^{(k)}_{\alpha_1}) \cdots \varphi(x^{(k)}_{\alpha_n}) \\ &\quad + [\varphi(x^{(k)}_{\alpha_1}) - \varphi(x^*_{\alpha_1})] \varphi(x^*_{\alpha_0}) \varphi(x^{(k)}_{\alpha_2}) \cdots \varphi(x^{(k)}_{\alpha_n}) + \cdots \\ &\quad + [\varphi(x^{(k)}_{\alpha_n}) - \varphi(x^*_{\alpha_n})] \varphi(x^*_{\alpha_0}) \cdots \varphi(x^*_{\alpha_{n-1}}) \Big\}, \end{aligned}$$

we obtain

$$\begin{aligned} |F(x^{(k)}) - F(x^*)| \le \frac{1}{n+1} \sum_{\alpha_0,\dots,\alpha_n=1}^{\infty} |a_{\alpha_0\cdots\alpha_n}| \Big[& |\varphi(x^{(k)}_{\alpha_0}) - \varphi(x^*_{\alpha_0})| |\varphi(x^{(k)}_{\alpha_1}) \cdots \varphi(x^{(k)}_{\alpha_n})| \\ & + |\varphi(x^{(k)}_{\alpha_1}) - \varphi(x^*_{\alpha_1})| |\varphi(x^*_{\alpha_0}) \varphi(x^{(k)}_{\alpha_2}) \cdots \varphi(x^{(k)}_{\alpha_n})| + \cdots \\ & + |\varphi(x^{(k)}_{\alpha_n}) - \varphi(x^*_{\alpha_n})| |\varphi(x^*_{\alpha_0}) \cdots \varphi(x^*_{\alpha_{n-1}})| \Big]. \end{aligned} \tag{3}$$

By virtue of the continuity of the function φ and the equality $\lim_{k\to\infty} x^{(k)}_{\alpha_j} = x^*_{\alpha_j}$, for arbitrary $\varepsilon > 0$ and sufficiently large k we shall have $|\varphi(x^{(k)}_{\alpha_j}) - \varphi(x^*_{\alpha_j})| < \varepsilon$ for all $j = 0, 1, \dots, n$. Therefore, from inequality (3), after some transformations we shall have

$$|F(x^{(k)}) - F(x^*)| \le \varepsilon M^{n+1} \left(\sum_{\alpha_0,\dots,\alpha_n=1}^{\infty} a^2_{\alpha_0\cdots\alpha_n} \right)^{1/2}.$$

Since ε is an arbitrary number, it follows from this and from (2) that $\lim_{k\to\infty} F(x^{(k)}) = F(x^*)$. Lemma 1 is proved.

Lemma 2. *The strong differential of the functional $F(x)$ generates the operator $L_F x$ with components*

$$L_{Fx} = \left(\varphi'(x_{\alpha_0}) \sum_{\alpha_1,\dots,\alpha_n=1}^{\infty} a_{\alpha_0\cdots\alpha_n} \varphi(x_{\alpha_1}) \cdots \varphi(x_{\alpha_n}) \right), \qquad \alpha_0 = 1, 2, \dots, \tag{4}$$

mapping elements $x \in S_1$ to elements l_2.

Indeed, let $x = (x_\alpha)$, $h = (h_\alpha) \in S_1$, and let t be a numerical operator, $x + th \in S_1$; then

$$\left.\frac{dF(x+th)}{dt}\right|_{t=0} = \frac{1}{n+1} \sum_{\alpha_0,\dots,\alpha_n=1}^{\infty} a_{\alpha_0\cdots\alpha_n} [h_{\alpha_0} \varphi'(x_{\alpha_0}) \varphi(x_{\alpha_1}) \cdots \varphi(x_{\alpha_n}) +$$

$$+ h_{\alpha_1} \varphi'(x_{\alpha_1}) \varphi(x_{\alpha_0}) \varphi(x_{\alpha_2}) \cdots \varphi(x_{\alpha_n}) + \cdots + h_{\alpha_n} \varphi'(x_{\alpha_n}) \varphi(x_{\alpha_n}) \cdots \varphi(x_{\alpha_{n-1}})].$$

Taking into account the symmetry of the coefficients $a_{\alpha_0\cdots\alpha_n}$, from this we obtain

$$\left.\frac{dF(x+th)}{dt}\right|_{t=0} = \sum_{\alpha_0=1}^{\infty} h_{\alpha_0} \varphi'(x_{\alpha_0}) \sum_{\alpha_1,\dots,\alpha_n=1}^{\infty} a_{\alpha_0\cdots\alpha_n} \varphi(x_{\alpha_1}) \cdots \varphi(x_{\alpha_n}). \tag{5}$$

The right-hand side of equality (5) is a linear functional with respect to h and is represented in the form of the scalar product

$$\left.\frac{dF(x+th)}{dt}\right|_{t=0} = (h, L_{Fx}), \tag{6}$$

where L_{Fx} is the operator with components (4). Let us note that

$$\varphi'(x_{\alpha_0})\sum_{\alpha_1,\dots,\alpha_n=1}^{\infty} a_{\alpha_0\cdots\alpha_n}\varphi(x_{\alpha_1})\cdots\varphi(x_{\alpha_n}) \le$$

$$\le \max_{x\subset S_1}\varphi'(x_{\alpha_0})M^n\left(\sum_{\alpha_1,\dots,\alpha_n=1}^{\infty} a^2_{\alpha_0\cdots\alpha_n}\right)^{1/2}<\infty.$$

This shows that the components L_F^x converge uniformly in the ball $x \in S_1$. Moreover, we have

$$\sum_{\alpha_0=1}^{\infty}\left[\varphi'(x_{\alpha_0})\sum_{\alpha_1,\dots,\alpha_n=1}^{\infty} a_{\alpha_0\cdots\alpha_n}\varphi(x_{\alpha_1})\cdots\varphi(x_{\alpha_n})\right]^2 \le$$

$$\le \max_{x\subset S_1}\varphi'^2(x_{\alpha_0})M^{2n}\sum_{\alpha_0,\dots,\alpha_n=1}^{\infty} a^2_{\alpha_0\cdots\alpha_n}<\infty,$$

which completes the proof of Lemma 2.

Lemma 3. The operator L_{Fx} satisfies a Lipschitz condition in S_1.

Let $x^{(1)} = (x^{(1)}_{\alpha_j})$, $x^{(2)} = (x^{(2)}_{\alpha_j})$, $\alpha_j = 1, 2, \dots$, be an arbitrary pair of elements of S_1. We represent the norm of the difference $L_F x^{(1)} - L_F x^{(2)}$ in the form

$$\|L_F x^{(1)} - L_F x^{(2)}\| = \left\{\sum_{\alpha_0=1}^{\infty}\left[\varphi'(x^{(1)}_{\alpha_0})\sum_{\alpha_1,\dots,\alpha_n=1}^{\infty} a_{\alpha_0\cdots\alpha_n}(\varphi(x^{(1)}_{\alpha_1})\cdots\varphi(x^{(1)}_{\alpha_n})-\right.\right.$$

$$\left.\left.-\varphi(x^{(2)}_{\alpha_1})\cdots\varphi(x^{(2)}_{\alpha_n})) + (\varphi'(x^{(1)}_{\alpha_0}) - \varphi'(x^{(2)}_{\alpha_0}))\sum_{\alpha_1,\dots,\alpha_n=1}^{\infty} a_{\alpha_0\cdots\alpha_n}\varphi(x^{(2)}_{\alpha_1})\cdots\varphi(x^{(2)}_{\alpha_n})\right]^2\right\}^{1/2}.$$

Using the equalities $\varphi(x^{(1)}_{\alpha_j}) - \varphi(x^{(2)}_{\alpha_j}) = (x^{(1)}_{\alpha_j} - x^{(2)}_{\alpha_j})\varphi'(\xi_1)$, where $j = 1, 2, \dots, n$, $-1 \le \xi_1 \le +1$, $\varphi'(x^{(1)}_{\alpha_0}) - \varphi'(x^{(2)}_{\alpha_0}) = (x^{(1)}_{\alpha_0} - x^{(2)}_{\alpha_0})\varphi''(\xi_2)$, where $-1 \le \xi_2 \le +1$, and the inequality $|x^{(1)}_{\alpha_0} - x^{(2)}_{\alpha_0}| \le \|x^{(1)} - x^{(2)}\|$, after some transformations we obtain

$$\|L_F x^{(1)} - L_F x^{(2)}\| \le 2M_1\left(\sum_{\alpha_0,\dots,\alpha_n=1}^{\infty} a^2_{\alpha_0\cdots\alpha_n}\right)^{1/2}\|x^{(1)} - x^{(2)}\|, \tag{7}$$

where

$$M_1 = \sup(nNM^{n-1}K;\ K_1M^n), \qquad K = \max\varphi'(\xi), \qquad K_1 = \max\varphi''(\xi), \qquad 0 \le \xi \le 1.$$

Lemma 4. L_{Fx} is weakly continuous in S_1.

Indeed, let $x^{(k)} \xrightarrow{\text{weak}} x$, where $x^{(k)} = (x^{(k)}_{\alpha_j})$, $x = (x_{\alpha_j})$, $\alpha_j = 1, 2, \ldots$, $j = 0, \ldots, n$; $x^{(k)}, x \in S_1$, $k = 1, 2, \ldots$; then $x^{(k)}_{\alpha_j} \to x_{\alpha_j}$ as $k \to \infty$ for all $j = 0, \ldots, n$. We shall have

$$\begin{aligned}\|L_F x^{(k)} - L_F x\| = \Bigg\{ \sum_{\alpha_0=1}^{\infty} \Big[(\varphi'(x^{(k)}_{\alpha_0}) - \varphi'(x_{\alpha_0})) & \sum_{\alpha_1,\ldots,\alpha_n=1}^{\infty} a_{\alpha_0\cdots\alpha_n} \varphi(x^{(k)}_{\alpha_1}) \cdots \varphi(x^{(k)}_{\alpha_n}) \\ & + \varphi'(x_{\alpha_0}) \sum_{\alpha_1,\ldots,\alpha_n=1}^{\infty} a_{\alpha_0\cdots\alpha_n} (\varphi(x^{(k)}_{\alpha_1}) - \varphi(x_{\alpha_1}))\varphi(x^{(k)}_{\alpha_2}) \cdots \varphi(x^{(k)}_{\alpha_n}) \\ & + (\varphi(x^{(k)}_{\alpha_2}) - \varphi(x_{\alpha_2}))\varphi(x_{\alpha_1})\varphi(x^{(k)}_{\alpha_3}) \cdots \varphi(x^{(k)}_{\alpha_n}) + \cdots \\ & + (\varphi(x^{(k)}_{\alpha_n}) - \varphi(x_{\alpha_n}))\varphi(x_{\alpha_1}) \cdots \varphi(x_{\alpha_{n-1}}) \Big]^2 \Bigg\}^{1/2}. \end{aligned} \tag{8}$$

By virtue of the continuity of φ and φ', for an arbitrary $\varepsilon > 0$ there exists a sufficiently large natural number N such that, for $k > N$, the inequalities

$$|\varphi(x^{(k)}_{\alpha_j}) - \varphi(x_{\alpha_j})| < \varepsilon, \quad |\varphi'(x^{(k)}_{\alpha_j}) - \varphi'(x_{\alpha_j})| < \varepsilon, \quad j = 0, 1, \ldots, n,$$

will hold; and from (8)

$$\|L_F x^{(k)} - L_F x\| \le \varepsilon c \left(\sum_{\alpha_0,\ldots,\alpha_n=1}^{\infty} a^2_{\alpha_0\cdots\alpha_n} \right)^{1/2}, \qquad c = 2\max(M^n, KnM^{n-1}).$$

Since ε is arbitrary, the validity of the lemma follows from this. From the weak continuity of the operator L_{Fx} and the regularity of the space l_2 follows the complete continuity of L_{Fx}.

Now suppose that $a_{\alpha_0\cdots\alpha_n} \ge 0$, φ is an even function, $\varphi > 0$ and $\varphi(0) = 0$; then the functional $F(x)$ is positive and even in S_1, and, consequently, L_F is odd. By virtue of Theorem 9 of paper $(^3)$, there exists an infinite system of geometrically distinct normalized eigen-elements $x^{(m)} = (x^{(m)}_{\alpha_j})$ satisfying the functional equation

$$L_F x^{(m)} = \lambda_m x^{(m)}, \qquad \|x^{(m)}\| = 1, \qquad \lambda_m = (x^{(m)}, L_F x^{(m)}), \qquad m = 1, 2, \ldots. \tag{9}$$

Equation (9) is equivalent to the following infinite system of equations with an infinite number of unknowns:

$$\varphi'(x_{\alpha_0}^{(m)}) \sum_{\alpha_1,\ldots,\alpha_n=1}^{\infty} a_{\alpha_0\cdots\alpha_n}\varphi(x_{\alpha_1}^{(m)})\cdots\varphi(x_{\alpha_n}^{(m)}) = \lambda_m x_{\alpha_0}^{(m)}, \qquad \alpha_0 = 1, 2, \ldots. \quad (9')$$

Thus the following theorem has been proved.

Theorem. *If a strongly differentiable functional* $F(x)$ *in the unit ball* $S_1 \subset l_2$ *is defined by equality (1), where* $a_{\alpha_0\cdots\alpha_n} \geq 0$ *are symmetric real coefficients,* φ *is a twice continuously differentiable even function on the segment* $[-1, +1]$, $\varphi > 0$, $\varphi(0) = 0$, *and if* $L_{Fx} = \operatorname{grad} F(x)$ *with components (4), then there exists an infinite system of geometrically distinct normalized eigen-elements* $x^{(m)} = (x_{\alpha_j}^{(m)})$, $m = 1, 2, \ldots$, *satisfying (9) (or* $(9')$*), where* λ_m *are the eigenvalues corresponding to the eigen-elements* $x^{(m)}$.

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Received
3 I 1957

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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