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Abstract

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CONDITIONS ON SHOCK WAVES IN THE GENERAL THEORY OF RELATIVITY

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Conditions on shock waves are derived for an ideal compressible medium in the general theory of relativity. It is shown for a centrally symmetric field that the components of the metric tensor of 4-space are continuous, while their derivatives are discontinuous upon passage through a shock wave. Conditions are given for the first and second derivatives of the components of the metric tensor on a shock wave in the general case.

1. Introduction. The motion of an ideal fluid in the general theory of relativity is described by the following system of equations: the equations of conservation of the total 4-momentum (^{1})

$$\frac{\partial}{\partial x^k}\left[(-g)(wu^iu^k - g^{ik}p + t^{ik})\right] = 0; \tag{1,1}$$

the equation of continuity

$$\frac{\partial}{\partial x^k}\left(\sqrt{-g}\rho u^k\right) = 0. \tag{1,2}$$

Here ρ is the rest density; p is the pressure; w is the heat function per unit proper volume; g^{ik} are the contravariant components of the metric tensor of 4-space with fundamental quadratic form $ds^2 = g_{ik}dx^idx^k$; $(-g) = |\operatorname{Det}(g_{ik})|$; t^{ik} is the pseudotensor of the energy-momentum density of the gravitational field; $u^i = dx^i/ds$ is the 4-velocity of a particle. Latin indices take the values $0, 1, 2, 3$; the Greek indices occurring below refer to spatial coordinates and take the values $1, 2, 3$; a twice-repeated index denotes summation.

Equations (1,1) are consequences of A. Einstein's gravitational equations, which we shall take in the form (^{1})

$$\frac{\partial h^{ikl}}{\partial x^l} = (-g)(T^{ik} + t^{ik}), \tag{1,3}$$

where

$$h^{ikl} = -h^{ilk} = \frac{1}{2\varkappa}\frac{\partial}{\partial x^m}\{(-g)(g^{ik}g^{lm} - g^{il}g^{km})\}; \tag{1,4}$$

T^{ik} is the energy-momentum tensor of matter; in the case of an ideal fluid it has the form

$$T^{ik} = wu^i u^k - g^{ik}p, \tag{1,5}$$

$\varkappa = 8\pi k/c^4$; k is Newton’ s gravitational constant; c is the speed of light in vacuum.

2. Conditions on a shock wave. Taking into account that

$$u^\alpha = u^0 v^\alpha, \qquad u^k\frac{\partial}{\partial x^k} = u^0\frac{d}{dt}, \qquad v^\alpha = \frac{dx^\alpha}{dt},$$

and using equation (1.2), system (1.1) can be given the form

$$\sqrt{-g}\,\rho u^0\frac{\partial}{\partial t}\left\{\frac{\sqrt{-g}}{\rho u_0}\left[wu^0u^i - g^{i0}p + t^{i0}\right]\right\} =$$

$$= \frac{\partial}{\partial x^\alpha}\left\{(-g)\left[p\left(g^{i\alpha} - g^{i0}v^\alpha\right) - t^{i\alpha} + t^{i0}v^\alpha\right]\right\}. \tag{2,1}$$

Instead of the differential equations (2.1) and (1.2), in deriving the conditions at discontinuities we shall take as the initial equations the integral equations

$$\frac{d}{dt}\int_{\tau_3^*}(-g)\left(wu^iu^0 - g^{i0}p + t^{i0}\right)d\tau =$$

$$= \int_\sigma\left[p\left(g^{i\alpha} - g^{i0}v^\alpha\right) - t^{i\alpha} + t^{i0}v^\alpha\right](-g)n_\alpha d\sigma, \tag{2,2}$$

$$\frac{d}{dt}\int_{\tau_3^*}\rho u^0\sqrt{-g}\,d\tau = 0, \tag{2,3}$$

where τ_3^* is a spatial fluid volume; $d\tau = dx^1dx^2dx^3$; σ is the surface bounding the volume τ_3^*; n_α is the vector of the outward normal to σ. Equations (2.2) and (2.3) are completely equivalent to equations (1.1) and (1.2) in a region where the integrands are continuously differentiable, but are more general in the sense that they retain their meaning also where the continuity of the integrands is violated, provided, of course, that an integrable discontinuity occurs.

We shall use L. I. Sedov’ s method for deriving conditions on a surface of strong discontinuity (^{2}). Let Σ be an isolated piecewise-smooth discontinuity surface, moving with normal velocity D and lying entirely inside the volume τ_3^* rigidly attached to it. At the time under consideration τ_3^* coincides with τ_3. For any function $A(t, x^1, x^2, x^3)$ the equality holds

$$\frac{d}{dt}\int_{\tau_3} A\,d\tau = \frac{d}{dt}\int_{\tau_3^*} A\,d\tau + \int_{\sigma} A\,(D_n - v^\alpha n_\alpha)\,d\sigma, \tag{2,4}$$

where D_n is the normal velocity on the surface σ of the volume τ_3. Noting that, when the volume τ_3 is contracted to the surface Σ, $\frac{d}{dt}\int_{\tau_3} A\,d\tau$ tends to zero uniformly with respect to t, we obtain, applying (2.4) to equations (2.2) and (2.3):

$$\left[(-g)\left\{wu^0u^i(D - v_n) - p\left(g^{i0}D - g^{i\alpha}n_\alpha\right) + t^{i0}D - t^{i\alpha}n_\alpha\right\}\right] = 0, \tag{2,5}$$

$$\left[\rho\sqrt{-g}\,u^0(D - v_n)\right] = 0. \tag{2,6}$$

Square brackets denote the jump of the quantities enclosed in them. If $D = 0$, $g^{00} = 1$, $g^{i\alpha} = -\delta^{i\alpha}$, then (2.5), (2.6) reduce to the equations of special relativity (^{3}):

$$[wu^0u^\alpha n_\alpha] = 0 \quad \text{—continuity of the energy flux through } \Sigma,$$

$$[wu^\beta u^\alpha n_\alpha - pn^\beta] = 0 \quad \text{—continuity of the momentum flux,}$$

$$[\rho u^\alpha n_\alpha] = 0 \quad \text{—continuity of the mass flux.}$$

3. Centrally symmetric field.

In this case, for

$$(x^0, x^1, x^2, x^3) = (t, r, \theta, \varphi)$$

$$g_{00} = e^{\nu(t,r)}, \quad g_{11} = -e^{\lambda(t,r)}, \quad g_{22} = -r^2, \quad g_{33} = -r^2\sin^2\theta, \quad g_{ij} = 0,\ i \neq j; \tag{3,1}$$

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right), \tag{3,2}$$

and the gravitational equations have the form (^{1})

$$-\varkappa T_1^1 = \varkappa \frac{\varepsilon v^2 e^{\lambda} + p e^{\nu}}{e^{\nu} - e^{\lambda} v^2} = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2},$$

$$-\varkappa T_2^2 = \varkappa p = \frac{1}{2} e^{-\lambda}\left(\nu'' + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\lambda'\nu'}{2}\right) - \frac{1}{2} e^{-\nu}\left(\ddot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\lambda}\dot{\nu}}{2}\right),$$

$$-\varkappa T_0^0 = -\varkappa \frac{\varepsilon e^{\nu} + p e^{\lambda} v^2}{e^{\nu} - e^{\lambda} v^2} = e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) - \frac{1}{r^2}, \tag{3,3}$$

$$-\varkappa T_0^1 = -\varkappa \frac{(p + \varepsilon) v e^{\nu}}{e^{\nu} - e^{\lambda} v^2} = e^{-\lambda}\frac{\dot{\lambda}}{r},$$

where $\varepsilon = w - p$ is the internal rest energy; $v = dr/dt$; a dot denotes differentiation with respect to t, and a prime with respect to r.

It is easy to see that system (3,3) is equivalent to the system

$$e^{\nu}\left(\nu'' + \frac{\nu'^2}{2} - \frac{\lambda'\nu'}{2}\right) - e^{\lambda}\left(\ddot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\lambda}\dot{\nu}}{2}\right) = \frac{2e^{\nu}}{r^2}(1 - e^{\lambda}) + \frac{a e^{\nu}}{r}, \tag{3,4}$$

$$v = \frac{e^{\nu - \lambda}}{2\dot{\lambda}}(a - \lambda' - \nu'), \qquad \varkappa p = \frac{e^{-\lambda}}{2r}\left\{\frac{2}{r}(1 - e^{\lambda}) + \nu' - \lambda' + a\right\}, \tag{3,5}$$

$$\varkappa \varepsilon = \frac{e^{-\lambda}}{2r}\left\{\frac{2}{r}(e^{\lambda} - 1) - \nu' + \lambda' + a\right\}, \qquad a = +\sqrt{(\lambda' + \nu')^2 - 4e^{\lambda - \nu}\dot{\lambda}^2}.$$

The nonzero components of the energy-momentum pseudotensor of the gravitational field are equal to

$$t^{00} = -\frac{1}{\varkappa}\frac{e^{-\nu}}{r}\left(\frac{5e^{-\lambda}}{r} + e^{-\lambda}\lambda' + \frac{\operatorname{ctg}^2\theta}{r}\right), \qquad t^{01} = \frac{1}{\varkappa} e^{-\nu - \lambda}\frac{\dot{\lambda}}{r},$$

$$t^{02} = \frac{1}{\varkappa} e^{-\nu}\frac{\dot{\lambda}}{r^2}\operatorname{ctg}\theta, \qquad t^{11} = -\frac{1}{\varkappa} e^{-2\lambda}\frac{1}{r}\left(\nu' - \frac{e^{\lambda}\operatorname{ctg}^2\theta}{r} + \frac{1}{r}\right), \tag{3,6}$$

$$t^{12} = -\frac{1}{\varkappa}\frac{e^{-\lambda}}{r^2}\operatorname{ctg}\theta\left(\frac{2}{r} + \nu'\right),$$

$$t^{22} = \frac{1}{2\varkappa r^2}\left(\frac{e^{-\lambda}}{2}\nu'\lambda' + 3\frac{e^{-\lambda}}{r} - e^{-\nu}\frac{\dot{\lambda}^2}{2} - e^{-\nu}\frac{\dot{\lambda}\dot{\nu}}{2} + e^{-\lambda}\frac{\nu'^2}{2} + e^{-\lambda}\frac{\lambda'}{r} + \frac{2e^{-\lambda}}{r^2}\right).$$

$$(-g) = e^{\lambda+\nu} r^4 \sin^2\theta. \tag{3,7}$$

Putting successively $i = 0, 1, 2$ in (2,5) and using equations (3,5), (3,6), (3,7), we obtain

$$\left[r^2(6\sin^2\theta + e^{\lambda}\cos 2\theta)D\right] = 0, \qquad [r^2 e^{\nu}\cos 2\theta] = 0, \tag{3,8}$$

$$[\dot{\lambda}D] + e^{\nu-\lambda}\nu' = 0. \tag{3,9}$$

It follows from (3,8) that the components of the metric tensor remain continuous in passing through the surface of a strong discontinuity. (3,9) imposes a condition on the derivatives of these components.

4. **The Cauchy problem.** We shall show that the values of the derivatives $\dot{\lambda}_{(2)}, \lambda'_{(2)}, \dot{\nu}_{(2)}, \nu'_{(2)}$ on one side of the discontinuity surface Σ, which we denote by (2), can, on the basis of relations (3,9) and (2,6), be expressed in terms of their values on the other side, which we denote by (1): $\dot{\lambda}_{(1)}, \lambda'_{(1)}, \dot{\nu}_{(1)}, \nu'_{(1)}$. In doing so, of course, one must use the equation of state of the medium to express $p = p(\rho, \varepsilon)$, and for p, ε, and v, use equations (3,5). Indeed, adding to (3,9) and (2,6) two strip conditions

$$\frac{d\lambda_0(t)}{dt} = \dot{\lambda}_{(2)} + D\lambda'_{(2)}, \qquad \frac{d\nu_0(t)}{dt} = \dot{\nu}_{(2)} + D\nu'_{(2)},$$

where $\lambda_0(t), \nu_0(t)$ are the values of λ and ν on the discontinuity surface Σ: $r = r_0(t)$, $D = dr_0/dt$, we obtain 4 equations for determining $\dot{\lambda}_{(2)}, \lambda'_{(2)}, \dot{\nu}_{(2)}, \nu'_{(2)}$.

This system has one trivial solution $\dot{\lambda}_{(2)} = \dot{\lambda}_{(1)}$, $\lambda'_{(2)} = \lambda'_{(1)}$, $\dot{\nu}_{(2)} = \dot{\nu}_{(1)}$, $\nu'_{(2)} = \nu'_{(1)}$, corresponding to a continuous change of the derivatives. If the motion of the fluid is known on one side of the discontinuity surface, then the problem of the motion on the other side is thereby reduced to the solution of two second-order equations (3,4) and (1,2) with respect to $\lambda(t, r)$, $\nu(t, r)$, with Cauchy data on Σ.

5. An arbitrary gravitational field. If from equations (1,3) and (1,2) and the equation of state of the medium one eliminates $p, \rho, \varepsilon, v^1, v^2, v^3$, then the problem reduces to solving 6 equations with respect to the 6 functions $g_{ij}(x^k)$, since 4 of them may be chosen arbitrarily. Here we assume that the functions

$g_{ij}(x^k)$ are continuous, while their derivatives may suffer integrable discontinuities on isolated hypersurfaces $x^0 = f(x^1, x^2, x^3)$. Replace (1,3) by the integral equations

$$\int_{S_3} h^{ikl} n_l \, dS_3 = \int_{\tau_4} (T^{ik} + t^{ik})(-g) \, d\tau_4, \tag{5,1}$$

where τ_4 is a 4-volume; S_3 is a smooth hypersurface bounding τ_4; n_l is the vector of the outward normal to S_3. If the discontinuity surface $\Sigma : \; x^0 = f(x^1, x^2, x^3)$ lies entirely inside τ_4, then, in view of the assumed integrability of the functions T^{ik} and t^{ik}, the integral over τ_4 in (5,1) tends to zero as S_3 is contracted to Σ, and we obtain

$$\left[h^{ikl} n_l\right]_\Sigma = 0. \tag{5,2}$$

Since (see (1,4))

$$h^{ikl} n_k n_l \big|_\Sigma \equiv 0,$$

there will be 6 independent relations (5,2). Then, if

$$g^0_{ij}(x^1, x^2, x^3) = g_{ij}(f(x^1, x^2, x^3), x^1, x^2, x^3),$$

then, adding to (5,2) the $6 \times 3 = 18$ first-order strip conditions

$$\frac{\partial g^0_{ij}}{\partial x^\alpha} = \frac{\partial g_{ij}}{\partial x^0} \frac{\partial f}{\partial x^\alpha} + \frac{\partial g_{ij}}{\partial x^\alpha}, \qquad \alpha = 1, 2, 3,$$

we obtain $18 + 6 = 24$ equations for determining the 24 first derivatives $\partial g_{ij}/\partial x^k$.

Since in the general case the discontinuity equation will contain third derivatives of the functions $g_{ij}(x^k)$, we must also specify conditions for the second derivatives, whose number is 60. These are:

1) the $9 \times 6 = 54$ second-order strip conditions

$$\frac{\partial g^0_{ij,k}}{\partial x^\alpha} = \frac{\partial g_{ij,k}}{\partial x^\alpha} + \frac{\partial g_{ij,k}}{\partial x^0} \frac{\partial f}{\partial x^\alpha}, \qquad \alpha = 1, 2, 3, \quad k = 0, 1, 2, 3,$$

where

$$g^0_{ij,k} = \frac{\partial g_{ij}}{\partial x^k}\bigg|_{x^0 = f(x^1, x^2, x^3)}, \qquad g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k};$$

2) 5 equations obtained from (1,3) after the elimination of $p, \varepsilon, v^1, v^2, v^3$;

3) condition (2,6).

Thus, in the general case as well we arrive at a problem with Cauchy data on the surface of a strong discontinuity.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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