

# LOCAL STRESSES IN THE TORSION OF A ROUND PRISMATIC BAR WITH AN ECCENTRIC ELLIPTICAL HOLE

1958

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Fig. 1

Figure 1: Fig. 1

**Abstract**

**Full Text**

**THEORY OF ELASTICITY**

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**LOCAL STRESSES IN THE TORSION OF A ROUND PRISMATIC BAR WITH AN ECCENTRIC ELLIPTICAL HOLE**

*(Presented by Academician N. I. Muskhelishvili, 25 XII 1957)*

Let us consider the problem of pure torsion of a round prismatic bar weakened in the longitudinal direction by an elliptical cavity; in this case the axis of the elliptical cavity is located at some distance from the axis of the solid bar (Fig. 1).

To determine the stressed state under pure torsion, one must find a complex function regular in the region  $S$  and satisfying the boundary conditions <sup>1</sup>

$$\varphi_1(t) + \overline{\varphi_1(t)} = 0 \quad \text{on } L_1; \tag{1}$$

$$\varphi_1(t) + \overline{\varphi_1(t)} = it\bar{t} + B \quad \text{on } L_2, \tag{2}$$

where  $t$  is the affix of a point of the circle  $L_1$  or  $L_2$ ;  $B$  is a certain real constant.

Fig. 1

Following the method of D. I. Sherman <sup>2</sup>, let us take a new function, regular everywhere outside the ellipse,

$$\varphi(z) = \varphi_1(z) - \frac{1}{2\pi i} \int_{L_1} \frac{\omega(t_1)}{t_1 - z} dt_1, \tag{3}$$

which vanishes at infinity. Using the function

$$z - e = A \left( \zeta + \frac{1}{\zeta} \right) \quad \left( A = \frac{\sqrt{a^2 - b^2}}{2}, \quad \rho = \sqrt{\frac{a+b}{a-b}} \right), \tag{4}$$

which conformally maps the exterior of the ellipse onto the exterior of the circle  $\gamma$  of radius  $\rho > 1$ , and using (3), after some transformations from (1), (2) we obtain

$$\begin{aligned} \varphi^*(\tau) + \overline{\varphi^*(\tau)} = & - \sum_{n=-\infty}^{\infty} b_n \left( \frac{\tau^n}{\rho^n} + \frac{\rho^n}{\tau^n} \right) + A^2 \left( \frac{\tau^2}{\rho^2} + \frac{\rho^2}{\tau^2} \right) + \\ & + Ae \left( \rho + \frac{1}{\rho} \right) \left( \frac{\tau}{\rho} + \frac{\rho}{\tau} \right) + A^2 \left( \rho^2 + \frac{1}{\rho^2} \right) + e^2 + B - (\alpha_0 + \overline{\alpha_0}), \end{aligned} \quad (5)$$

where  $\varphi^*(\zeta) = \varphi(z)$ ;

$$\begin{aligned} b_n = \frac{1}{\rho^n} \sum_{k=|n|}^{\infty*} \left( \frac{A}{R-e} \right)^k C_k^{\frac{n+k}{2}} \alpha_k \quad (n = \pm 1, \pm 2, \dots); \\ b_0 = \sum_{k=2}^{\infty*} \left( \frac{A}{R-e} \right)^k C_k^{\frac{k}{2}} \alpha_k; \end{aligned} \quad (6)$$

besides <sup>3</sup>,

$$b_{-n} = \rho^{2n} b_n.$$

Here

$$\alpha_k = \frac{(R-e)^k}{2\pi i} \int_{L_1} \frac{\omega(t_1)}{(t_1-e)^{k+1}} dt_1 \quad (k = 0, 1, \dots). \quad (7)$$

The functional  $\alpha_0$  is, generally speaking, a complex quantity; we shall agree to regard the remaining constants  $\alpha_k$  ( $k = 1, 2, \dots$ ) as real. Below we shall verify the validity of this assumption. The asterisk on the summation sign indicates that the index  $n$  assumes values of the same parity as the values assumed by the index  $k$ .

Taking into account that  $\varphi^*(\zeta)$  is regular outside the circle of radius  $\rho$  in the  $\zeta$ -plane and vanishes at infinity, from (5) we find

$$\varphi^*(\zeta) = \sum_{n=1}^{\infty} \lambda_n \frac{\rho^n}{\zeta^n}; \quad B = -A^2 \left( \rho^2 + \frac{1}{\rho^2} \right) - e^2 + (\alpha_0 + \overline{\alpha_0}) + 2b_0. \quad (8)$$

Here

$$\lambda_n = -(1 + \rho^{2n})b_n + A^2 \delta_n^{(2)} + Ae \left( \rho + \frac{1}{\rho} \right) \delta_n^{(1)}; \quad (9)$$

$$\delta_n^{(1)} = \begin{cases} 1 & (n = 1), \\ 0 & (n \neq 1), \end{cases} \quad \delta_n^{(2)} = \begin{cases} 1 & (n = 2), \\ 0 & (n \neq 2). \end{cases}$$

Using the function

$$\zeta = \frac{z - e + \sqrt{(z - e)^2 - 4A^2}}{2A},$$

inverse to the function (4), where the plus sign is taken before the radical, since under the mapping an infinitely distant point in the  $\zeta$ -plane must pass into an infinitely distant point of the  $z$ -plane and the value of the radical is taken positive for positive  $z > e + 2A$ , from (8) we obtain

$$\varphi(z) = \sum_{n=1}^{\infty} \lambda_n \left( \frac{\rho}{2A} \right)^n \left[ (z - e) - \sqrt{(z - e)^2 - 4A^2} \right]^n. \quad (10)$$

On the basis of the auxiliary imaginary function introduced on the circle  $L_1$ ,

$$2\omega(t) = \varphi_1(t) - \overline{\varphi_1(t)},$$

and the function (3), we have

$$\omega(t) = \varphi(t) - \overline{\varphi(t)} + \alpha_0^*. \quad (11)$$

Here  $\alpha_0^*$  is a purely imaginary constant,

$$\alpha_0^* = \frac{1}{2\pi i} \int_{L_1} \omega(t_1) \frac{dt_1}{t_1}.$$

Taking into account expression (7) and adopting the notation

$$\beta_m = \frac{R^m}{2\pi i} \int_{L_1} \frac{\omega(t)}{t^{m+1}} dt, \quad (12)$$

from (10), after some transformations and reasoning, we obtain

$$\beta_m = \sum_{n=1}^m \rho^n \lambda_n J_{m,n}, \quad (13)$$

where

$$J_{m,n} = -\left(\frac{e}{R}\right)^m \sum_{\nu_1=n-1}^{N(m,n)} * \left(\frac{A}{e}\right)^{\nu_1+1} C_{n-1}^{\nu_1} \left(C_{\nu_1}^{\frac{\nu_1-n+1}{2}} - C_{\nu_1}^{\frac{\nu_1-n-1}{2}}\right),$$

with  $N(m, n) = m - 1$  if  $m$  and  $n$  are of the same parity, and  $N(m, n) = m - 2$  if  $m$  and  $n$  are of different parity.

By virtue of (12), the  $\beta_{k_1}$  are the coefficients of the expansion of the function  $\omega(t)$  in a Fourier series; therefore for it we have

$$\omega(t) = \beta_0 + \sum_{k_1=1}^{\infty} \beta_{k_1} \left[ \left(\frac{t}{R}\right)^{k_1} - \left(\frac{R}{t}\right)^{k_1} \right]. \quad (14)$$

Substituting the expression for the function  $\omega(t)$  into formula (7), we obtain

$$x_k = \left(\frac{R}{e} - 1\right)^k \sum_{k_1=k}^{\infty} (-1)^{k_1-k} C_{-k-1}^{k_1-k} \left(\frac{e}{R}\right)^{k_1} \beta_{k_1} \quad (k = 0, 1, 2, \dots). \quad (15)$$

On the basis of formulas (6), (15), from (9), after interchanging the order of summation, we shall have

$$\lambda_n = \frac{1}{\rho^n} \sum_{k_1=n}^{\infty} \mu_{k_1,n} \beta_{k_1} + A^2 \mathcal{G}_n^{(2)} + Ae \left(\rho + \frac{1}{\rho}\right) \mathcal{G}_n^{(1)}. \quad (16)$$

From formulas (16) and (13), after interchanging the order of summation, we finally obtain

$$\sum_{k_1=1}^{\infty} q_{k_1,m} \beta_{k_1} = f_m \quad (m = 1, 2, \dots); \quad (17)$$

where

$$\begin{aligned} q_{m,m} &= 1 - \sum_{n=1}^m J_{m,n} \mu_{m,n}, & q_{k_1,m} &= - \sum_{n=1}^{E(k_1,m)} J_{m,n} \mu_{k_1,n}, & f_m &= \\ & & &= Ae(1 + \rho^2) J_{m,1} + A^2 \rho^2 J_{m,2}, \end{aligned}$$

with  $E(k_1, m) = k_1$ , if  $k_1 \leq m$ , and  $E(k_1, m) = m$ , if  $k_1 > m$ .

**Fig. 2**

Fig. 2

Figure 2: Fig. 2

Fig. 3

Figure 3: Fig. 3

**Fig. 3**

From the system of equations (17), for  $a/b = 5$ ,  $e/k = 0.375$ , and  $A/e = \frac{4}{25}\sqrt{6}$ , the first 10 equations were solved by the method of successive approximations; moreover, it was necessary to find only 4 approximations in order to arrive at a quite accurate solution (see Table 1).

**Table 1**

| $k_1$ | $-\frac{1}{A^2}\beta_{k_1}$ | $k_1$ | $-\frac{1}{A^2}\beta_{k_1}$ |
|-------|-----------------------------|-------|-----------------------------|
| 1     | 1.0165347                   | 6     | $0.2634644 \cdot 10^{-1}$   |
| 2     | 0.4147684                   | 7     | $0.1460082 \cdot 10^{-1}$   |
| 3     | 0.1901036                   | 8     | $0.8264082 \cdot 10^{-2}$   |
| 4     | $0.939431 \cdot 10^{-1}$    | 9     | $0.4756865 \cdot 10^{-2}$   |
| 5     | $0.488524 \cdot 10^{-1}$    | 10    | $0.2775966 \cdot 10^{-2}$   |

**Table 2**

| $z$  | $\delta/A^2$             | $z$      | $\Delta, \%$        |
|------|--------------------------|----------|---------------------|
| $R$  | $0.411221 \cdot 10^{-2}$ | $e + a$  | $0.4 \cdot 10^{-6}$ |
| $iR$ | $0.1469 \cdot 10^{-2}$   | $e + ib$ | $0.2 \cdot 10^{-6}$ |
| $-R$ | $0.20562 \cdot 10^{-2}$  | $e - a$  | $0.3 \cdot 10^{-6}$ |

On the basis of (3), (10), and (14), in the region  $S$  we obtain

$$\varphi_1(z) = \sum_{k=1}^{\infty} \left\{ \lambda_k \left( \frac{\rho}{2A} \right)^k [z - e - \sqrt{(z - e)^2 - 4A^2}]^k + \beta_k \left( \frac{z}{R} \right)^k \right\} + \beta_0. \quad (18)$$

Taking into account that the critical points are determined by the affixes  $z_{1,2} = e \pm 2A$ , to expression (18), at any point of the region under consideration, we shall further give

has the form (see Fig. 2)

$$\varphi_1(z) = \sum_{k=1}^{\infty} \left\{ \lambda_k \left( \frac{\rho}{2A} \right)^k \left[ z - e - \sqrt{BE \cdot BD} e^{i \frac{\theta_1 + \theta_2}{2}} \right]^k + \beta_k \left( \frac{z}{R} \right)^k \right\} + \beta_0,$$

where

$$BE = \sqrt{|z|^2 - (e + 2A)(z + \bar{z}) + (e + 2A)^2}, \quad \cos \theta_1 = \frac{z + \bar{z} - 2(e + 2A)}{2BE},$$

$$BD = \sqrt{|z|^2 - (e - 2A)(z + \bar{z}) + (e - 2A)^2}, \quad \cos \theta_2 = \frac{z + \bar{z} - 2(e - 2A)}{2BD}.$$

The quantities  $\delta = \varphi_1 + \bar{\varphi}_1$  on  $L_1$ ,  $\Delta = \frac{\varphi_1 + \bar{\varphi}_1 - \bar{t} - B}{t + B} 100\%$  on  $L_2$  (for the example considered,  $B = -9.6468014 A^2$ ), which characterize the degree of accuracy with which the function  $\varphi_1(z)$  found satisfies the boundary conditions at the characteristic points, are given in Table 2. From the table it is seen that at the points of the ellipse the boundary condition is satisfied exactly, and at the points of the circle sufficiently accurately.

**Table 3**

| $z$                   | $Y_z \frac{R^3}{M}$ | $Y_z^{(0)} \frac{R^3}{M}$ | $\frac{Y_z}{Y_z^{(0)}}$ |
|-----------------------|---------------------|---------------------------|-------------------------|
| $R$                   | 0.703               | 0.637                     | 1.1                     |
| $\frac{R + e + a}{2}$ | 0.657               | 0.533                     | 1.23                    |
| $e + a$               | 2.166               | 0.430                     | 5.04                    |
| $e - a$               | 0.839               | 0.048                     | 17.62                   |
| $-R$                  | -0.652              | -0.637                    | 1.02                    |

The torsional rigidity is determined by the formula

$$D = \mu(I + D_0), \quad (19)$$

where  $\mu$  is the shear modulus;  $I$  is the polar moment of inertia of the area of the cross-section of the bar:

$$I = \pi \left[ \frac{R^4}{2} - ab \left( \frac{a^2 + b^2}{4} + e^2 \right) \right], \quad D_0 = -\frac{i}{4} \sum_{k=1}^2 \int_{L_k} [\varphi_1(t) - \bar{\varphi}_1(\bar{t})] dt. \quad (20)$$

From formula (19), on the basis of formulas (20), we obtain

$$D = \pi\mu \left\{ \frac{R^4}{2} - ab \left( \frac{a^2 + b^2}{4} + e^2 \right) - A \left( 2A\lambda_2 + \left( \rho + \frac{1}{\rho} \right) \lambda_1 e \right. \right. \\ \left. \left. + \sum_{k=1}^{\infty} \beta_k \left( \frac{e}{R} \right)^\nu \left[ \frac{2A}{\rho^2} \sum_{\nu=0}^{Q_1(k)} \left( \frac{A}{e} \right)^\nu C_k^\nu \left( C_\nu^{\frac{\nu+2}{2}} - \rho^4 C_\nu^{\frac{\nu-2}{2}} \right) \right. \right. \\ \left. \left. + e \left( 1 + \frac{1}{\rho^2} \right) \sum_{\nu=1}^{Q_2(k)} \left( \frac{A}{e} \right)^\nu C_k^\nu \left( C_\nu^{\frac{\nu+1}{2}} - \rho^2 C_\nu^{\frac{\nu-1}{2}} \right) \right] \right\}.$$

Here  $Q_1(k) = k$ , if  $k$  is even;  $Q_1(k) = k - 1$ , if  $k$  is odd;  $Q_2(k) = k - 1$ , if  $k$  is even;  $Q_2(k) = k$ , if  $k$  is odd.

For the example considered ( $a/R = 0.3$ ,  $b/R = 0.06$ ),  $D = 1.5155 \mu R^4$ .

The calculated values of the stresses at points of the cross-section situated on the  $x$ -axis are given in Table 3. Also given there are the values of the stresses  $Y_z^{(0)}$ , when the section is a circle without weakening. As is seen from the table, at the right and left vertices of the ellipse the stresses  $Y_z$  are respectively 5.04 and 17.6 times greater than the stresses  $Y_z^{(0)}$  at the same points.

The epure of the tangential stresses acting at points of the  $x$ -axis is shown in Fig. 3.

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Received  
24 XII 1957

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*Note: Figure translations are in progress. See original paper for figures.*

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