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Abstract

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SOME PROBLEMS FOR THE EULER-POISSON-DARBOUX EQUATION

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From the works (^{1,2}) it is known that every real analytic, in the variables x and y, solution $w(x, y)$ of the equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{c}{y}\frac{\partial w}{\partial y} = 0, \qquad c = \text{const}, \tag{1}$$

defined in a domain containing within it a segment of the line of singularity $y = 0$ $(c \neq 0, -1, -2, \ldots)$, is uniquely determined by its values for $y = 0$.

In the present paper, using the definitions and notation of work (^{5}), for equation (1) we consider the following three problems with data on the line $y = 0$.

Problem C_1. On the interval L a function $f_1(x)$ is given. In the domain T, find a solution $w(x, y) \in E_c(T)$ $(c \geq 1)$ satisfying on L the condition

$$w(x, 0) = f_1(x). \tag{2}$$

Problem C_2. On the interval L two functions $f_1(x)$ and $f_2(x)$ are given. In the domain T, find a solution $w(x, y) \in E_c(T)$ $(-1 < c < 0,\ 0 < c < 1)$ satisfying on L condition (2) and the condition

$$\lim_{y \to 0} y^c \frac{\partial w}{\partial y} = f_2(x). \tag{3}$$

Problem C_3. On the interval L two functions $f_1(x)$ and $f_2(x)$ are given. In the domain $D = T \cup L \cup \overline{T}$, find a solution $w(x, y) \in E_c(D)$ $(-1 < c < 0,\ 0 < c < 1)$, satisfying on L conditions (2) and (3).

Theorem 1. *A solution of problem C_1 for equation (1) with $c \geq 1$ in the domain T exists if and only if there exists a function $\varphi(z)$, analytic in T, which takes on L the real values $f_1(x)$.*

The sufficiency of the theorem follows from the representation of solutions of equation (1) for $c \geq 1$ considered in work (^{3}), and the necessity—from the possibility proved there of representing any solution $w(x, y) \in E_c(T)$ $(c \geq 1)$ by means of an analytic function $\varphi(z)$ in a domain of class B, and from work (^{6})—in an arbitrary domain.

Theorem 2. *For the existence of a solution of problem C_2 for equation (1) with $(-1 < c < 0,\ 0 < c < 1)$ in the domain T, it is sufficient that at least one of the following conditions be fulfilled:*

a) the existence of two functions $\varphi(z)$ and $\psi(z)$, analytic in T, which take on L, respectively, the real values

$$\varphi(x) = f_1(x); \tag{4}$$

$$\psi(x) = \left(\frac{1}{1-c}\right)^c f_2(x); \tag{5}$$

b) the existence in T of the conjugate solution and the existence of a function $\varphi(z)$, analytic in T, satisfying condition (4) on L;

c) the existence in T of the conjugate solution and the existence of a function $\psi(z)$, analytic in T, satisfying condition (5) on L.

The proof of the theorem in the case of item a) follows directly from the representations considered in papers (^{4,5}). In the case of item b), by means of the function $\varphi(z)$ in T we construct the first term $w_1(x, y)$ of the indicated representations, which satisfies on L the conditions:

$$w_1(x, 0) = \varphi(x); \tag{6}$$

$$\lim_{y \to 0} y^c \frac{\partial w_1}{\partial y} = 0. \tag{7}$$

Then, by means of the conjugate solution $w^*(x, y)$, we construct the desired solution of problem C_2 in the form $w(x, y) = 2w_1(x, y) - w^*(x, y)$. Case c) of the theorem is considered analogously.

Corollary. If each of the values (4) and (5) on L is real and analytic, then there exists a domain T', adjacent to L, in which a solution of problem C_2 exists.

Theorem 3. *For the existence of a solution of problem C_3 in the domain $D = T \cup L \cup \widetilde{T}$ for equation (1), under $(-1 < c < 0,\ 0 < c < 1)$, condition a) of Theorem 2 is necessary and sufficient.*

Sufficiency follows directly from the representations of solutions in the domain D considered in papers (^{4,5}); necessity follows from the uniqueness, proved there, of the corresponding representation in D of any solution $w(x, y) \in E_c(D)$, $D = T \cup L \cup \widetilde{T}$, and from the existence of functions $\varphi(z)$ and $\psi(z)$, analytic in T, satisfying on L conditions (4) and (5). Directly from papers ($^{3-5}$) it follows that the solutions of problems C_1, C_2, and C_3 are unique.

Let us consider the question of the correctness of the solutions of problems C_1, C_2, and C_3. A subdomain belonging to T and adjacent to the segment $L' \cup \{a, b\} \subset L$ will be denoted by T', its closure by $\widetilde{T}'$, and the closure of $T' \cup L' \cup \widetilde{T}'$ will be denoted by $\widetilde{D}'$.

Solutions $w(x, y)$ of problem C_1, belonging to the class $\{w(x, y)\}$, are correct in T if, for any $E > 0$ and for any $\widetilde{T}' \subset T$, there exists an $m > 0$ such that, for any $w(x, y) \in \{w(x, y)\}$ satisfying on L the condition

$$|w(x, 0)| < m, \tag{8}$$

the inequality

$$|w(x, y)| < E \tag{9}$$

holds for all points of $\widetilde{T}'$.

Solutions $w(x, y)$ of problem C_2, belonging to the class $\{w(x, y)\}$, are correct in T if, for any $E > 0$ and for any $\widetilde{T}' \subset T \cup L$, there exists an $m > 0$ such that, for any $w(x, y) \in \{w(x, y)\}$ satisfying on L condition (8) and the condition

$$\lim_{y \to 0} \left(\frac{|y|}{1 - c} \right)^c \left| \frac{\partial w}{\partial y} \right| < m, \tag{10}$$

the inequality (9) holds for all points of $\widetilde{T}'$.

Solutions $w(x, y)$ of problem C_3, belonging to the class $\{w(x, y)\}$, are correct in $D = T \cup L \cup \widetilde{T}$ if, for any $E > 0$ and for any $\widetilde{T}' \subset T \cup L$, there exists–

there is such an $m > 0$ that, for any $w(x, y) \in \{w(x, y)\}$ satisfying on L the conditions (8) and (10), the inequality (9) holds for all points of $\widetilde{D}'$.

We shall further assume that $w(x, y) \in E_c^*(T)$, $T \in B$, for $-1 < c < 0$, and $w(x, y) \in E_c(T)$ for $c > 0$.

Lemma 1. If the family of solutions $\{w(x, y)\}$ and the family of solutions conjugate to them $\{w^*(x, y)\}$ are uniformly bounded in T, then the corresponding families of functions $\{\varphi(z)\}$ and $\{\psi(z)\}$, from the representations (3) of (4) and (9) of (5), are uniformly bounded in any domain $\widetilde{T}' \subset T \cup L$.

Proof. Since for each $w(x, y)$ from $\{w(x, y)\}$ there exists a conjugate solution $w^*(x, y)$, $w(x, y)$ is represented in T as the sum of two terms:

$$w(x, y) = w_1(x, y) + w_2(x, y)$$

(for $c \geq 1$, $w_2(x, y) \equiv 0$), while the corresponding $w^*(x, y)$ from $\{w^*(x, y)\}$ is represented in the form

$$w(x, y) = w_1(x, y) - w_2(x, y).$$

Therefore each of the families $\{w_1(x, y)\}$ and $\{w_2(x, y)\}$ is bounded in T.

To prove the boundedness of the family $\{\varphi(z)\}$ in any $\widetilde{T}' \subset T \cup L$, suppose the contrary: suppose that, although the family $\{w_1(x, y)\}$ is uniformly bounded in T, the family $\{\varphi(z)\}$ is unbounded in some $\widetilde{T}' \subset T \cup L$. This means that there will be found a sequence of functions $\varphi_n(z)$ $(n = 1, 2, ...)$ from $\{\varphi(z)\}$ and a sequence of points z_n $(n = 1, 2, ...)$ in $\widetilde{T}'$ such that, as $n \to \infty$, $|\varphi_n(z_n)|$ tends monotonically to infinity. The sequence of points z_n in $\widetilde{T}'$ has at least one point of accumulation α. The sequence of solutions $w_{1n}(x, y)$ $(n = 1, 2, ...)$, corresponding to the functions $\varphi_n(z)$ $(n = 1, 2, ...)$, is bounded in T and compact in any $\widetilde{T}' \subset T \cup L$. Therefore in T there exists a subsequence of solutions $w_{1n_k}(x, y)$ $(k = 1, 2, ...)$ converging to a function $w_{10}(x, y)$, defined in T, which has zero limit (7), i.e. $w_{10}(x, y) \in N_c(T)$. The latter indicates that in T there exists a certain analytic function $\varphi_0(z)$ satisfying on L the condition

$$\varphi_0(x) = w_{10}(x, y).$$

The subsequence of analytic functions in T, $\varphi_{n_k}(z)$, corresponding to w_{1n_k} $(k = 1, 2, ...)$, in view of the monotonicity of $|\varphi_n(z)|$ at the points z_n, tends to infinity in a neighborhood of the point α. Moreover, since on L

$$\varphi_{n_k}(z) = w_{1n_k}(x, 0) \qquad (k = 1, 2, ...),$$

$\varphi_{n_k}(x)$ on L converges to $\varphi_0(x)$.

By the correctness of the Cauchy problem for solutions of the Laplace equation (7), we obtain that for any E and any $\widetilde{T}' \subset T \cup L$ there is an $m > 0$ such that, for any $\varphi_{n_k}(z)$ satisfying on L the condition

$$|\varphi_{n_k}(x) - \varphi_0(x)| < m,$$

the inequality

$$|\varphi_{n_k}(z) - \varphi_0(z)| < E$$

holds for all points of $\widetilde{T}'$. Since $\varphi_0(z)$ is defined in T and bounded in any domain $\widetilde{T}^* \subset T \cup L$, it follows that, beginning with some $k_0 > 0$, for all $k > k_0$ the functions $\varphi_{n_k}(z)$ will be bounded in $\widetilde{T}'$, and hence also at the point α. The latter contradicts the earlier assumption; therefore the family $\{\varphi(z)\}$ is uniformly bounded in $\widetilde{T}'$.

The boundedness of the family $\{\psi(z)\}$ in any domain $\widetilde{T}' \subset T \cup L$ is proved analogously. The lemma is proved.

Theorem 4. The solutions $w(x, y)$ of problem C_1 in the domain T for equation (1), for $c \geq 1$, are correct in the class of bounded solutions.

Theorem 5. The solutions $w(x, y)$ of problem C_2 in the domain T for equation (1), for $(-1 < c < 0,\ 0 < c < 1)$, are correct in the class of solutions possessing conjugate solutions and uniformly bounded together with their conjugate solutions.

Theorem 6. *The solutions $w(x, y)$ of problem C_3 in the domain D for equation (1), when $(-1 < c < 1)$ and $(0 < c < 1)$, are well posed in the class of bounded solutions.*

Since the proofs of these theorems are analogous, we give only the proof of Theorem 5.

In the domain T, by the preceding lemma, the collection of bounded solutions generates two families of analytic functions, $\{\varphi(z)\}$ and $\{\psi(z)\}$, bounded in any domain $\widetilde{T}' \subset T \cup L$.

It follows from work (^{7}) that such families are well posed in T, i.e., for any $E_1 > 0$ and any domain $\widetilde{T}' \subset T \cup L$ there exists an $m > 0$ such that, for all functions $\varphi(z)$ from $\{\varphi(z)\}$ and $\psi(z)$ from $\{\psi(z)\}$ satisfying on L the conditions

$$|\varphi(x)| < m, \qquad |\psi(x)| < m,$$

the conditions $|\operatorname{Re}\varphi(z)| < E_1$ and $|\operatorname{Re}\psi(z)| < E_1$ are fulfilled for all z in the domain $\widetilde{T}'$. Therefore, on the basis of the representations of papers ($^{3-6}$), for all points of the domain $\widetilde{T}'$, when $0 < c < 1$, we have the inequality:

$$w(x, y) < E_1 \left[1 + \left(\frac{d}{1-c} \right)^{1-c} \right]$$

(d is the diameter of the domain T). For values of c from the interval $-1 < c < 0$, along with the proved boundedness of the family $\{\operatorname{Re}\varphi(z)\}$ in $\widetilde{T}'$, according to (^{7}) we have uniform boundedness of the values of the gradient $\{\operatorname{grad}\operatorname{Re}\varphi(z)\}$, and hence also uniform boundedness of the derivative $|d\varphi/dz| < K$.

Hence, from representation (9) of paper (^{5}), for all points of the domain $\widetilde{T}'$ it is true that

$$|w(x, y)| < \frac{d}{1+c} K + E_1 + \left(\frac{d}{1-c} \right)^{1-c} E_1.$$

Choosing

$$E_1 = \frac{E}{1 + \left(\frac{d}{1-c}\right)^{1-c}} \qquad \text{for } 0 < c < 1;$$

$$E_1 = \frac{E + \left(\frac{d}{1+c}\right) K}{1 + \left(\frac{d}{1-c}\right)^{1-c}} \qquad \text{for } -1 < c < 0,$$

we complete the proof of the theorem.

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Note: Figure translations are in progress. See original paper for figures.