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**Abstract**

**Full Text**

**I. I. Danilyuk**

**On Mappings Corresponding to Solutions of Equations of Elliptic Type**

*(Presented by Academician M. A. Lavrent'ev, 16 XII 1957)*

1. Below we consider certain geometric properties of regular solutions of a system of first-order differential equations belonging to the elliptic type and reduced to the form\*

$$u_x - v_y = a(x, y)u + b(x, y)v, \quad u_y + v_x = c(x, y)u + d(x, y)v, \quad (1)$$

as well as elliptic equations of the second order

$$A_1(x, y) \frac{\partial^2 U}{\partial x^2} + 2B_1(x, y) \frac{\partial^2 U}{\partial x \partial y} + C_1(x, y) \frac{\partial^2 U}{\partial y^2} + D_1(x, y) \frac{\partial U}{\partial x} + E_1(x, y) \frac{\partial U}{\partial y} + F_1(x, y)U = 0. \quad (*)$$

The system (1) is equivalent to a single equation in complex notation:

$$\frac{\partial f}{\partial \bar{z}} = A(x, y)\bar{f} + B(x, y)\bar{z},$$

$$4A = (a - d) + i(c + b), \quad 4B = (a + d) + i(c - b), \quad (2)$$

if we introduce the notation  $z = x + iy$ ,  $2\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$ ,  $f = u + iv$ .

As examples show <sup>(1)</sup>, mappings corresponding to differential systems of the form (1) may be interior (in the sense of Stoilow), but may also differ sharply from interior ones. On the other hand, with the aid of the means and methods developed in the works of I. N. Vekua <sup>(2,3)</sup>, one can construct solutions of a general system of the form (1) giving interior mappings of a sufficiently small neighborhood of the point  $z = 0$ . Using these methods, we shall show in the present note that, under certain assumptions on the coefficients  $a(x, y), \dots, d(x, y)$  of the system (1), the latter has, in any (simply connected) domain, solutions that effect interior mappings of it onto certain Riemann surfaces. An analogue of this theorem also holds for equation (\*).

**2.** By means of a conformal mapping, the case of an arbitrary domain is reduced to the case of the unit disk  $K$ . Suppose that the functions  $a(x, y), \dots, d(x, y)$  depend analytically on the real arguments  $x, y$  in the closed disk  $\bar{K}$ . Extend the functions  $A(x, y), B(x, y)$  to the domain of complex values of  $x, y$  and make the substitution  $z = x + iy, \zeta = x - iy$ , so that  $z = \bar{\zeta}$  when  $x, y$  are real. Suppose that we obtain functions  $A(z, \zeta), B(z, \zeta)$ , analytic in the closed unit bicylinder  $\bar{K}_z \times \bar{K}_\zeta$  in the two complex arguments  $z, \zeta$ , and coinciding with  $A(x, y), B(x, y)$  on the plane  $\Sigma : z = \bar{\zeta}$ . Denoting here and below the analytic function  $f(z, \bar{z})$  by  $f^*(z, \zeta)$ , we replace equation (2) by the more general one:

$$\frac{\partial f(z, \zeta)}{\partial \zeta} = A(z, \zeta)f(z, \zeta) + B(z, \zeta)f^*(\zeta, z), \quad (3)$$

which coincides with the original one on the same plane  $\Sigma$  (2).

\* One auxiliary quasiconformal mapping also reduces to the form (1) a system whose differential part is the Beltrami system.

**Theorem 1.** Under the assumptions made, there always exist solutions  $f = u + iv$  of equation (2) which realize interior mappings of the unit disk onto a certain Riemann surface.

**Proof.** First of all, by means of the substitution

$$f(z, \zeta) = F(z, \zeta)a(z, \zeta), \quad a(z, \zeta) = \exp \left\{ \int A(z, \zeta) d\zeta \right\}, \quad (4)$$

where  $\alpha = \int A(z, \zeta) d\zeta$  denotes some primitive of  $A(z, \zeta)$  with respect to  $\zeta$ , equation (3) is reduced to the simpler form

$$\frac{\partial F(z, \zeta)}{\partial \zeta} = C(z, \zeta)F^*(\zeta, z), \quad C(z, \zeta) = B(z, \zeta) \exp \left\{ \int A^*(\zeta, z) dz - \int A(z, \zeta) d\zeta \right\}. \quad (5)$$

As was shown in [2], § 10, the general representation of all regular solutions of equation (5) is given by the formula

$$F(z, \zeta) = \varphi(z) + \int_{z_1}^z \Gamma_1(z, \zeta, t, \zeta_0) \varphi(t) dt + \int_{\zeta_1}^\zeta \Gamma_2(z, \zeta, z_0, \tau) \varphi^*(\tau) d\tau, \quad (6)$$

where  $z_0, z_1$  are arbitrary fixed points of  $K_z$ ;  $\zeta_0, \zeta_1$  are arbitrary fixed points of  $K_\zeta$ ;  $\Gamma_1(z, \zeta, t, \tau), \Gamma_2(z, \zeta, t, \tau)$  are the so-called resolvents of equation (5), holomorphic in all their arguments;  $z, t \in K_z$ ;  $\zeta, \tau \in K_\zeta$ ; and  $\varphi(z)$  is an arbitrary holomorphic function. From our assumptions it follows that in the closed bicylinder  $\bar{K}_z \times \bar{K}_\zeta, \Gamma_1, \Gamma_2$ , as well as their partial derivatives, are analytic and

continuous. If  $\varphi(z)$  is an arbitrary function holomorphic in  $K_z$ , then the function  $f(z, \bar{z})$ , where  $f(z, \zeta)$  is defined by formulas (4), (6), satisfies equation (2) in the disk  $K_z$ .

The idea of the proof is to ensure, by means of a suitable choice of the function  $\varphi(z)$ , the interior character of the mapping  $w = F(z, \bar{z})$ , where  $F(z, \zeta)$  is determined by (6). A sufficient condition for this is that the Jacobian  $J$  of the mapping not vanish inside  $K$ , which, as is easy to verify, can be written, taking equation (5) into account, in the form

$$J(z) = \left| \frac{\partial F}{\partial z} \right|^2 - \left| \frac{\partial F}{\partial \bar{z}} \right|^2 = J_1 \left( \left| \frac{\partial F}{\partial z} \right| + \left| \frac{\partial F}{\partial \bar{z}} \right| \right); \quad J_1 = \left( \left| \frac{\partial F}{\partial z} \right| - |C(z, \bar{z})| |F| \right). \quad (7)$$

Let us examine the difference  $J_1$ . Since, in the case of analytic functions, the substitution  $\zeta = \bar{z}$  and differentiation with respect to  $\zeta$  are interchangeable operations, this difference, by formula (6), can be written in the form

$$J_1(z) = \omega_1(z) - \omega_2(z),$$

where

$$J_1 = \left\{ \left| \varphi'(z) + \Gamma_1(z, \bar{z}, z, \zeta_0) \varphi(z) + \int_{z_0}^z \frac{\partial \Gamma_1(z, \bar{z}, t, \zeta_0)}{\partial z} \varphi(t) dt + \int_{\zeta_0}^{\bar{z}} \frac{\partial \Gamma_2(z, \bar{z}, z_0, \tau)}{\partial z} \varphi^*(\tau) d\tau \right| - |C(z, \bar{z})| \left| \varphi(z) + \int_{z_0}^z \Gamma_1(z, \bar{z}, t, \zeta_0) \varphi(t) dt + \int_{z_0}^{\bar{z}} \Gamma_2(z, \bar{z}, z_0, \tau) \varphi^*(\tau) d\tau \right| \right\}. \quad (8)$$

The subsequent transformations consist in estimating the minuend  $\omega_1(z)$  from below and the subtrahend  $\omega_2(z)$  from above. For this purpose put  $z_0 = -1$ ,  $\zeta_0 = -1$ , and first estimate one of the integrals entering into  $\omega_1$ . Assuming that integration is carried out along the path consisting of the segment of the real axis  $[-1, x]$  and the vertical segment joining the points  $(x, 0)$  and  $(x, y)$ , we obtain

$$\begin{aligned} & \left| \int_{-1}^z \frac{\partial \Gamma_1(z, \bar{z}, t, -1)}{\partial z} \varphi(t) dt \right| \leq \\ & \leq \left| \int_{-1}^x \frac{\partial \Gamma_1(z, \bar{z}, t, -1)}{\partial z} \varphi(t) dx \right| + \left| \int_0^y \frac{\partial \Gamma_1(z, \bar{z}, t, -1)}{\partial z} \varphi(t) dy \right| \leq \end{aligned}$$

$$\leq \left( \int_{-1}^x \left| \frac{\partial \Gamma_1}{\partial z} \right|^2 dx \right)^{1/2} \left( \int_{-1}^x |\varphi(t)|^2 dx \right)^{1/2} + \left( \int_0^y \left| \frac{\partial \Gamma_1}{\partial z} \right|^2 dy \right)^{1/2} \left( \int_0^y |\varphi(t)|^2 dy \right)^{1/2}, \quad (9)$$

if we use Bunyakovsky's inequality. Put now

$$\varphi(z) = e^{Nz}, \quad (10)$$

where  $N$  is a certain positive numerical parameter, whose value we shall choose subsequently. Then from inequality (9) we obtain ( $\varkappa$  is a certain constant):

$$\begin{aligned} \left| \int_{-1}^z \frac{\partial \Gamma_1(z, \bar{z}, t, -1)}{\partial z} \varphi(t) dt \right| &\leq \varkappa \left[ \left( \int_{-1}^x e^{2Nx} dx \right)^{1/2} + \left( \int_0^y e^{2Nx} dy \right)^{1/2} \right] = \\ &= \varkappa \left[ \frac{1}{2N} (e^{2Nx} - e^{-2N})^{1/2} + e^{Nx} |y|^{1/2} \right] = \\ &= \varkappa e^{Nx} \left[ \frac{1}{2N} (1 - e^{-2N(1+x)})^{1/2} + |y|^{1/2} \right] \equiv e^{Nx} \mu_1(N, x, y), \quad (11) \end{aligned}$$

where the function  $\mu_1(N, x, y)$  defined by the last identity, for sufficiently large  $N$ , is uniformly bounded in the closed unit disk, since  $1 + x \geq 0$ . The other integrals entering into formula (8) are estimated analogously, so that we have

$$\left| \int_{-1}^z \frac{\partial \Gamma_2(z, \bar{z}, -1, \tau)}{\partial z} \varphi^*(\tau) d\tau \right| \leq e^{Nx} \mu_2(N, x, y), \quad (12)$$

where  $\mu_2(N, x, y)$  possesses properties analogous to those of  $\mu_1$ . Denoting further

$$\varkappa_1 = \max_{z \in \bar{K}} |\Gamma_1(z, \bar{z}, -1)|$$

and using (10), (11), (12), we obtain:

$$\omega_1(z) \geq e^{Nx} |N - \{\varkappa_1 + \mu_1(N, x, y) + \mu_2(N, x, y)\}|. \quad (13)$$

By virtue of the noted properties of the functions  $\mu_1, \mu_2$ , it is clear that, choosing the parameter  $N$  sufficiently large, we shall satisfy the inequality

$$N - \{\nu_1 + \mu_1(N, x, y) + \mu_2(N, x, y)\} > 0 \quad (14)$$

throughout the unit disk  $\overline{K}$ .

Carrying out an analogous upper estimate for the subtrahend  $\omega_2$  entering into formula (8), we obtain

$$\omega_2(z) \leq Me^{Nx} \{1 + \mu_3(N, x, y) + \mu_4(N, x, y)\}, \quad (15)$$

where  $M = \max_{z \in \overline{K}} |C(z, \bar{z})|$ , and the functions  $\mu_3, \mu_4$ , in their properties, are analogous to the functions  $\mu_1, \mu_2$ .

Thus, increasing, if necessary, the parameter  $N$  once more, we shall ensure that the function  $N - (\nu_1 + \mu_1 + \mu_2) - M(1 + \mu_3 + \mu_4)$  is positive in the closed disk  $\overline{K}$ , and from (8), (14), (15) we obtain

$$J_1(z) \equiv \omega_1(z) - \omega_2(z) \geq e^{Nx} \{N - (\nu_1 + \mu_1 + \mu_2) - M(1 + \mu_3 + \mu_4)\}.$$

Thus, the difference  $J_1$ , and with it also the Jacobian (7), will be positive everywhere in  $K$ , so that the mappings  $w = F(z, \bar{z})$ , where  $F(z, \zeta)$ , constructed by formulas (6), (10), will be interior mappings if the parameter  $N$  exceeds a certain value.

We now return to formula (4) and prove the theorem in the general case.

Calculating by formula (7) the Jacobian of the mapping  $w = f(z, \bar{z})$ , we obtain

$$J_2(z) = \left( \left| \frac{\partial f}{\partial z} \right| + \left| \frac{\partial f}{\partial \bar{z}} \right| \right) \left( \left| \frac{\partial F}{\partial z} + F \frac{\partial a}{\partial z} \right| |a| - |AaF + \bar{B}\bar{a}\bar{F}| \right), \quad (16)$$

if we take equation (2) into account. As above, the matter reduces to estimating the difference entering (16). The minuend is estimated from below:

$$\begin{aligned} |a| \left| \frac{\partial F}{\partial z} + F \frac{\partial a}{\partial z} \right| &\geq |a| \left\{ \left| \frac{\partial F}{\partial z} \right| - \left| \frac{\partial a}{\partial z} \right| |F| \right\} \geq \\ &\geq |a| e^{Nx} \{N - (\tilde{\chi}_1 + \tilde{\mu}_1 + \tilde{\mu}_2) - \tilde{M}(1 + \tilde{\mu}_3 + \tilde{\mu}_4)\}, \end{aligned} \quad (17)$$

similarly to the difference entering formula (7);  $\tilde{M}, \tilde{\chi}_1$  are certain constants depending only on the coefficients of the system (1);  $\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3, \tilde{\mu}_4$  are analogous in their properties to the function  $\mu_1$ . The subtrahend in (16) is estimated from above:

$$|AaF + \bar{B}\bar{a}\bar{F}| \leq |a| M_1 |F| \leq |a| M_1 e^{Nx} \{1 + \tilde{\mu}_3 + \tilde{\mu}_4\}, \quad (18)$$

where  $M_1$  is a certain constant, analogously to the function  $\omega_2(z)$ . And since  $|a| \geq a_0 > 0$ ,  $a_0 = \text{const}$ , by virtue of definition (4), it follows from the estimates (17), (18), as before, that the Jacobian  $J_2(z)$  in the closed unit disk is strictly positive if the number  $N$  is taken sufficiently large. Theorem 1 is completely proved\*.

3. **Theorem 2.** *If the coefficients of equation (1) are analytic in the real arguments  $x, y$  in a closed simply connected domain  $G$  and can be continued as analytic functions of  $z$  and  $\zeta$  in the bicylinder  $G_z \times G_\zeta$ , then there always exists a complex-valued solution  $w = f(z, \bar{z})$  of it which realizes an interior mapping of the domain  $G$  onto some Riemann surface.\**
4. The circumstances by virtue of which the system (1) has interior mappings, as well as the appearance of "folds" or even of complete degeneration under mappings corresponding to systems of the form (1), can be explained geometrically from the following general point of view.

From equation (3) we first draw the following conclusion: in order that the continuation  $f(z, \zeta)$  of any regular solution  $f(x, y)$  of equation (2) with coefficients  $A, B$  analytic in  $x, y$ \*\* into the domain of complex arguments  $z, \zeta$  depend on only one complex variable  $z$ , it is necessary and sufficient that the system (1) be the Cauchy–Riemann system.

Thus, if we are not dealing with this distinguished case of the system (1), then, in studying the mappings corresponding to it, we are studying, in essence, the mapping of the plane  $\Sigma : z = \bar{\zeta}$  in the space of complex variables  $z, \zeta$  by means of functions  $f(z, \zeta)$  of two complex variables satisfying equation (3), and the image of any neighborhood  $U$  in the  $z$ -plane under the mapping by means of any solution  $f$  of the system (1) is the projection of a certain analytic surface  $U' = f(U)$  in the four-dimensional space  $(w, w_1)$  onto the coordinate  $w$ -plane.

In conclusion I express my sincere gratitude to I. N. Vekua for his attention to the work and valuable advice.

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2. I. N. Vekua, *Mat. sbornik*, **31(73)**, 2 (1952).
3. I. N. Vekua, *New Methods for Solving Elliptic Equations*, M.–L., 1948.

\* Under the same assumptions, Theorem 1 is also valid for general elliptic systems of the first order.

\*\* In this case the solution  $f(x, y)$  is also analytic in  $x, y$  ((2), § 10).

*Note: Figure translations are in progress. See original paper for figures.*

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