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# MATHEMATICS

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## Abstract

## Full Text

MATHEMATICS

L. N. PROKOPENKO

# THE CAUCHY PROBLEM FOR AN EQUATION OF S. L. SOBOLEV TYPE

*(Presented by Academician S. L. Sobolev on 7 VI 1958)*

In the paper [1], S. L. Sobolev investigated the Cauchy problem for partial differential equations of the form

$$\frac{d^m}{dt^m}Au + \sum_{k=0}^{m-1} B_k t \frac{d^k}{dt^k} u = f(t). \tag{1}$$

Similar equations have been considered by a number of authors ([2–4]). M. I. Vishik ([5]) considered an operator equation of the form (1) under the assumption that the operator $A^{-1}$ is bounded. In the present note, equation (1) is considered without this latter restriction. The main attention is devoted to finding those operators $B_k$ for which equation (1), for a given $A$, is solvable.

1°. Consider in a Hilbert space $\mathcal{H}$ equation (1) with the initial conditions

$$\left.\frac{d^k u}{dt^k}\right|_{t=0} = u_k \quad (k = 0, 1, \dots, m-1). \tag{2}$$

Suppose that the following conditions are satisfied:

1) The operators $A$ and $B_k(t)$ are closed in the Hilbert space $\mathcal{H}$, with everywhere dense domains of definition; $B_k(t)$ depend strongly continuously on $t$, $t \geq 0$, and for each $t$, $\mathfrak{D}(A) \subseteq \mathfrak{D}(B_k(t))$.
2) The operator $A$ has an inverse $A^{-1}$ (not necessarily continuous), defined on the everywhere dense set $\mathfrak{D}(A^{-1}) = \mathfrak{R}(A)$ in $\mathcal{H}$.
3) The operators $B_k(t)A^{-1}$ and $A^{-1}B_k(t)$ are defined, at least, on $\mathfrak{D}(A^{-1})$ and $\mathfrak{D}(A)$, respectively, and are bounded; and the operators $\overline{B_k(t)A^{-1}}$ and $\overline{A^{-1}B_k(t)}$, obtained after their closure to all of $\mathcal{H}$, are weakly continuous in $t$.
4) For all $t \geq 0$, $f(t) \in \mathfrak{D}(A^{-1})$, and the vector-functions $f(t)$ and $A^{-1}f(t)$ are weakly continuous in $t$.

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Under these assumptions one can prove the following lemma:

**Lemma.** *Let conditions 1)–4) be satisfied, and let the initial data* $u_k$ *belong to* $\mathfrak{D}(A)$, $k = 0, 1, \dots, m-1$. *There exists a vector-function* $u(t)$, $t \geq 0$, *weakly differentiable with respect to* $t$ *up to order* $m$ *inclusive, belonging for each* $t \geq 0$, *together with the indicated derivatives, to* $\mathfrak{D}(A)$, *satisfying equation (1) and the initial conditions (2). This solution is unique and stable in the sense that the inequality*

$$\|u(t)\| \leq c(t) \left\{ \sum_{k=0}^{m-1} \|u_k\| + \sup_{0 \leq \tau \leq t} \left\| A^{-1} f(\tau) \right\| \right\} \tag{*}$$

*holds, with continuous* $c(t) > 0$.

Let us briefly outline the proof. By the usual device, equation (1) is reduced to a system having the form of equation (1) with $m = 1$ and with initial condition $u(0) = u_0$, and for the resulting equation the conditions of the theorem are satisfied. By the change of unknown function $Au = v$, it is reduced to the equation

$$\frac{dv}{dt} = \overline{B(t)A^{-1}}\, v + f$$

with initial condition $v(0) = Au_0$. In view of the continuity of the operator $\overline{B(t)A^{-1}}$, this equation has a solution for every $u_0 \in \mathfrak{D}(A)$, obtained by the method of successive approximations:

$$v(t) = \lim_{n\to\infty} v_n(t), \qquad v_n(t) = Au_0 + \int_0^t \overline{B(\tau)A^{-1}}\, v_{n-1}(\tau)\, d\tau + \int_0^t f(\tau)\, d\tau. \tag{3}$$

However, in general this solution does not belong to $\mathfrak{D}(A^{-1})$ if $A^{-1}$ is an unbounded operator, and therefore from it one cannot obtain a solution $u(t)$ of equation (1). Therefore, in the case of unboundedness of $A^{-1}$, the boundedness of the operator $A^{-1}B$ proves to be essential. Using this, from (3) we obtain

$$A^{-1}v_n(t) = u_0 + \int_0^t \overline{A^{-1}B(\tau)}\, A^{-1}v_{n-1}(\tau)\, d\tau + \int_0^t A^{-1}f(\tau)\, d\tau,$$

i.e. $A^{-1}v_n$ are the successive approximations for the equation

$$u(t) = u_0 + \int_0^t \overline{A^{-1}B(\tau)}\, u(\tau)\, d\tau + \int_0^t A^{-1}f(\tau)\, d\tau$$

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and, consequently, converge to the solution $u(t)$ of this equation: $A^{-1}v_n \to u$. Hence, from (3), by virtue of the closedness of $A^{-1}$, it follows that $v \in \mathfrak{D}(A^{-1})$ and $u(t) = A^{-1}v(t)$ is a solution of equation (1).

2°. Let $L[u]$ be a differential expression of second order

$$L[u] = \sum_{\alpha,\beta=1}^{n} \alpha_{\alpha\beta}(x)\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + \sum_{\gamma=1}^{n} a_\gamma(x)\frac{\partial u}{\partial x_\gamma} + a(x)u \tag{4}$$

with complex coefficients, continuously differentiable throughout the $n$-dimensional space $E_n$ as many times as is the order of the corresponding derivative. By $L^+$ we denote the differential expression formally adjoint to $L$, with passage to the complex-conjugate coefficients. From $L$ and $L^+$ we construct in $L_2(E_n) = \mathcal{H}$ the operators $B$ and $B_0^+$ with $\mathfrak{D}(B_0) = \mathfrak{D}(B_0^+) = C_0^\infty$ ($C_0^\infty$ is the set of infinitely differentiable functions with compact supports). It is not difficult to see that $B_0^+ \subseteq B_0^*$, $B_0 \subseteq B_0^{+*}$, and therefore the operators $B_0$ and $B_0^+$ admit closures $B$ and $B^+$. In particular, by $A$ we denote the analogous operator constructed from the $n$-dimensional Laplace operator taken with the minus sign. $A$ is a self-adjoint operator for which zero is a point of the continuous spectrum, and therefore $A^{-1}$ exists but is not bounded.

Let us see under what conditions on the coefficients of $L$ the conditions of the lemma will be satisfied for the operators $A$ and $B$. The boundedness of the operator $BA^{-1}$, as is not difficult to see, is equivalent to the inequality $\|Bu\| \leq c\|Au\|$, holding for $u \in C_0^\infty$, while the boundedness of the operator $A^{-1}B$ is equivalent to the analogous inequal-

to the inequality $\|B^\tau u\| \leq c\|Au\|$. In turn, these relations follow from the following conditions imposed on the coefficients of $L$:

A. $\quad D^m a_{\alpha\beta} \in S_m, \quad m = 0,1,2; \qquad D^m a_\gamma \in S_{m+1}, \quad m = 0,1; \qquad a \in S_2$

($D^m$ denotes $m$-fold differentiation). Here $S_m$ denotes the class of locally summable functions $\varphi(x)$ in $E_n$ satisfying the inequality

$$\int |\varphi(x)u(x)|^2\, dx \leq C_\varphi(A^k u, u) \tag{5}$$

for all $u \in C_0^\infty$. Consequently, if conditions A are fulfilled for the coefficients of the differential expressions generating the operators $B_k$, and condition 4) is fulfilled for the free term, equation (1) is solvable.

However, generally speaking, this solution is a generalized one. In order to obtain a classical solution, one may use the following device: multiply equation (1) on the left by $A^p$ and assume that the conditions of the lemma are satisfied for the equation so obtained. In addition, one must require the boundedness of the operator $A^{p-1}BA^{-p}$, corresponding to the operator $BA^{-1}$, which is again equivalent to the fulfillment of the inequality

$$\|A^{p-1}Bu\| \le c\|Au\|,$$

and this latter is a consequence of the following conditions on the coefficients of $L$:

A'. $D^m a_{\alpha\beta} \in S_m$; $D^m a_\gamma \in S_{m+1}$; $D^m a \in S_{m+2}$; $m = 1, 2, \dots, 2(p-1)$,

under which (assuming the existence of all the indicated derivatives) the solution $u(t)$ of equation (1) will belong, for every $t \ge 0$, to

$$\mathfrak{D}(A^p) = W_2^{(2p)}(E_n)$$

($W_2^{(2p)}(E_n)$ is the Sobolev space of functions possessing in $E_n$ generalized derivatives up to order $2p$ that are square summable over the whole space; see [6]). For $p > \frac{1}{2}n + 2$, as follows from the Sobolev embedding theorem [6], $u(x,t)$ will be twice continuously differentiable with respect to the spatial variables and $m$ times continuously differentiable with respect to time and, consequently, will be a classical solution of the equation (in this case it is natural to assume that the initial data $u_k \in \mathfrak{D}(A^p)$ and that the free term satisfies condition 4) with $f$ replaced by $A^{p-1}f$ and $A$ by $A^p$).

**Remark.** The equation (1) is treated in an analogous way also in the case where the coefficients $L$ depend on $t$. In this case it is additionally necessary to require, first, uniform continuity of all the functions entering into relations A and A' ; second, local boundedness in $t$ of the constants occurring on the right-hand side of the inequalities (5) for the functions indicated above.

3°. We shall now describe the classes $S_m$. One can prove the following assertion:

If $\varphi(x)$ is a bounded measurable function, $\varphi(x) = O(|x|^{-q})$ as $|x| \to \infty$, where

$$q > m' = \max\left\{m, \frac{4m-n}{2}\right\},$$

and $q \ge 0$ for $m = 0$, then $\varphi \in S_m$, $m = 0, 1, \dots$.

In proving this assertion one uses the properties of fundamental solutions of the polyharmonic equation. Together with what has been set out above, the last fact makes it possible to formulate the following theorem:

**Theorem.** Let the coefficients of the differential expressions $L_k(t)$ of the form (4) be differentiable up to order $2(p-1)$ inclusive (up to the order equal to the order of the corresponding derivative when $p = 1$), uniformly continuous in $t$ together with the indicated derivatives, and satisfy conditions

$$\left|D^m a_{\alpha\beta}^{(k)}\right| \leq C(1+|x|)^{-(m'+\varepsilon_m)}; \qquad \left|D^m a_{\gamma}^{(k)}\right| \leq C(1+|x|)^{-(m+1)'+\varepsilon_{m+1}};$$

$$\left|D^m a^{(k)}\right| \leq C(1+|x|)^{-(m+2)'+\varepsilon_{m+2}}$$

uniformly in $t$ on every finite interval. Suppose, in addition, that the free term satisfies condition 4) with $f(t)$ replaced by $A^{p-1}(t)$, and $u_k \in \mathfrak{D}(A^p)$. There exists a function $u(x,t)$, $m$ times (weakly) differentiable with respect to $t$, belonging together with its derivatives to $\mathfrak{D}(A^p) = W_2^{(2p)}(E_n)$, satisfying equation (1), the initial conditions (2), and inequality (*). For $p > \frac{1}{2}n + 2$ this solution is twice continuously differentiable with respect to $x$, $m$ times continuously differentiable with respect to $t$, and is a classical solution of equation (1).

**Remark.** The theorem imposes higher requirements on the coefficients at the lower derivatives with respect to the rate of decrease at infinity; this is closely connected with the unboundedness of $A^{-1}$. In particular, coefficients (in $x$) constant for $L$ can occur only at the second derivatives.

$4^0$. In conclusion we add that other operators $A$ may be considered analogously; however, in doing so serious difficulties arise in describing the classes $S_k$: in order to obtain nontrivial characteristics of these classes, a detailed investigation of the structure of the operator $A$ is needed, for example, the study of fundamental solutions in the case of their existence. In this case one can obtain a fundamental solution for equation (1), analogous to that obtained in (1), through which the solution $u(t)$ of problem (1)–(2) is represented. Moreover, all the constructions described can be carried out in other spaces, which makes it possible to extend the stock of initial data.

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*Note: Figure translations are in progress. See original paper for figures.*

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