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Abstract

Full Text

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ON A METHOD FOR SOLVING INTEGRAL EQUATIONS

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Following an analogy with the theory of linear operators ([1,2]), instead of Fredholm integral equations of the second kind

$$\varphi(x) = \lambda \int_a^b K(x,s)\varphi(s)\,ds + f(x) \tag{1}$$

and of the first kind

$$\int_a^b K(x,s)\varphi(s)\,ds = f(x) \tag{2}$$

we shall consider a single integral equation of the form

$$\int_a^b K(x,s)\varphi(s)\,ds = \mu\varphi(x) + F(x), \tag{3}$$

where $K(x,s)$ and $\varphi(x)$ are the same as in (1) and (2), i.e., the kernel and the unknown function; $F(x)$ is a function given on the interval $[a,b]$ $(b > a)$; μ is a numerical parameter. By the **eigenvalues** of the kernel $K(x,s)$ we mean those values μ for which equation (3), with $F(x) \equiv 0$, has a nonzero solution. This nonzero solution is called the **eigenfunction** of the kernel $K(x,s)$ corresponding to the eigenvalue μ.

All functions connected with the integral equation will be assumed summable, and the integrals (in the sense of Lebesgue)

$$C^2 = \int_a^b |K(x,s)|^2\,ds, \qquad B^2 = \int_a^b \int_a^b |K(x,s)|^2\,dx\,ds,$$

$$D^2 = \int_a^b |F(x)|^2\,dx, \qquad E^2 = \int_a^b |\varphi(x)|^2\,dx \tag{4}$$

will be considered finite.

Theorem 1. *If the kernel $K(x,s)$ is real and symmetric, and the number μ, different from zero, is not an eigenvalue of the kernel $K(x,s)$, then the sequence of functions*

$$\widetilde{\varphi}_{m+1}(x) = -\frac{1}{\mu}F(x) + \psi_{m+1}(x) \qquad (m = 0, 1, ...), \tag{5}$$

where the functions $\psi_{m+1}(x)$ are defined by the substitution formula

$$\psi_{m+1}(x) = P\psi_m + \frac{2}{\sigma}F^*(x) \quad (m = 0, 1, 2, ...), \qquad \psi_0(x) = \frac{2}{\sigma}F^*(x), \tag{6}$$

$$P\psi = \psi(x) - \frac{2}{\sigma}\left\{\int_a^b N(x,s)\psi(s)\,ds + \mu^2\psi(x)\right\}; \tag{7}$$

$$N(x,s) = K_2(x,s) - 2\mu K(x,s); \tag{8}$$

$$K_2(x,s) = \int_a^b K(x,t)K(t,s)\,dt; \tag{9}$$

$$F^*(x) = \int_a^b \left[K(x,s) + \frac{1}{\mu}N(x,s)\right] F(s)\,ds; \tag{10}$$

σ is a number satisfying the inequality

$$\sigma > (B + |\mu|)^2, \tag{11}$$

converges uniformly as $m \to \infty$ to the exact solution of integral equation (3).

Theorem 2. If the kernel $K(x,s)$ is real and symmetric, and the number $\mu = 0$ is not an eigenvalue of the kernel $K(x,s)$, then the sequence of functions defined by the substitution formula

$$\widetilde{\varphi}_{m+1}(x) = G\widetilde{\varphi}_m + \frac{2}{\sigma}F^*(x) \quad (m = 0, 1, 2, ...), \tag{12}$$

where $\widetilde{\varphi}_0(x)$ is an arbitrary function;

$$G\varphi = \varphi(x) - \frac{2}{\sigma}\int_a^b K_2(x,s)\varphi(s)\,ds, \tag{13}$$

$$F^*(x) = \int_a^b K(x,s)F(s)\,ds; \tag{14}$$

σ is a number satisfying the inequality

$$\sigma > B^2, \tag{15}$$

converges in the mean as $m \to \infty$ to the exact solution of equation (3) for $\mu = 0$.

Theorem $1'$**.** If $K(x,s)$ is a complex symmetric kernel, and the real parameter μ, different from zero, is not an eigenvalue of the kernel $K(x,s)$, then the sequence of functions

$$\widetilde{\varphi}_{m+1}(x) = -\frac{1}{\mu}F(x) + \psi_{m+1}(x) \quad (m = 0, 1, 2, ...), \tag{5'}$$

where the functions $\psi_{m+1}(x)$ are defined by the substitution formula (6), converges uniformly as $m \to \infty$ to the exact solution of equation (3).

Theorem $2'$**.** If $K(x,s)$ is a complex symmetric kernel, and the number $\mu = 0$ is not an eigenvalue of the kernel $K(x,s)$, then the sequence of functions $\widetilde{\varphi}_{m+1}(x)$, defined by formula (12), converges in the mean as $m \to \infty$ to the exact solution of equation (3) for $\mu = 0$.

Let now, in equation (3), the parameter μ, different from zero, be a real number (the latter does not restrict generality), and let the kernel $K(x,s)$ be any real or complex kernel. Then, if μ is not an eigenvalue of the kernel $K(x,s)$, the solution of integral equation (3) can be obtained, with the aid of the substitutions constructed above, as the solution

of the integral equation with symmetric kernel of the form

$$\int_a^b N(x,s)\varphi(s)\,ds = \mu^2\varphi(x) + \mu F(x) - \int_a^b \overline{K(s,x)}F(s)\,ds, \tag{16}$$

where

$$N(x,s) = \mu\,[\overline{K(s,x)}K(x,s)] - {}'K(x,s); \tag{17}$$

$${}'K(x,s) = \int_a^b \overline{K(t,x)}K(t,s)\,dt \tag{18}$$

(the bar above denotes the complex conjugate quantity).

Next, let in equation (3) with an arbitrary real or complex kernel $K(x,s)$ the parameter $\mu = 0$. Then, if $\mu = 0$ is not an eigenvalue of the kernel $K(x,s)$, as well as of the kernel $\overline{K(x,s)}$, the solution of integral equation (3) is obtained by the substitution formula (12) as the solution of an integral equation with symmetric kernel of the form

$$\int_a^b{}' K(x,s)\varphi(s)\,ds = \int_a^b \overline{K(s,x)}F(s)\,ds. \tag{19}$$

Thus, in all cases in which the parameter μ in equation (3) does not coincide with an eigenvalue of the kernel $K(x,s)$, the exact or approximate solution of this integral equation is obtained by the method of substitutions indicated here. These substitutions, as can be shown, lead to an expansion of the solution of the corresponding Fredholm integral equation of the second kind in a series in eigenfunctions, something which earlier, under direct consideration of the Fredholm equation of the second kind, could not be obtained, while the indicated expansion in eigenfunctions was of a purely existential nature ([3–7]).

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named after T. G. Shevchenko

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Note: Figure translations are in progress. See original paper for figures.