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Abstract

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MATHEMATICS

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On the problem with an oblique derivative for elliptic systems of first order

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1. Let G be a domain in the plane $z = x + iy$, and let $\Gamma = \sum_{j=0}^{m} \Gamma_j$ be the boundary of G, consisting of simple, closed, mutually nonintersecting H-continuous curves, with Γ_0 containing all the others inside it. In the domain G consider the equation

$$\frac{\partial f(z)}{\partial \bar{z}} = B(z)\overline{f(z)}, \qquad f = u + iv, \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{1}$$

equivalent to a system of two real equations in u, v. The present paper is devoted to the following problem.

Problem B. Find a complex-valued function $f = u + iv$, continuous in $G + \Gamma$, which is a generalized solution of equation (1) in the domain G, has a derivative $\partial f/\partial z$ continuously extendable to Γ, and satisfies on Γ the boundary condition

$$\operatorname{Re}\left[a\frac{\partial f}{\partial z} + bf\right] = \gamma, \qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \tag{2}$$

where a, b, and γ are prescribed functions on Γ.

Problem B reduces to a boundary-value problem for a general elliptic system of first order in the plane with a boundary condition of the form $\alpha u_x + \beta u_y + \gamma v_x + \delta v_y + \nu u + \mu v = g$; for harmonic functions it was first posed by Poincaré. In the class of analytic functions, in the general case this problem was studied in the work of I. N. Vekua ([4]), where necessary and sufficient conditions for its solvability were established in terms of the corresponding singular integral equation equivalent to the problem. In the book ([5]) it is studied for an equation of second order. Problem B was studied directly in ([3]), where the adjoint problem was introduced; in its terms necessary and sufficient conditions for solvability were given, and the relation of Theorem 4 was established (see below). In the present paper another approach to these questions is proposed.

The principal means of studying Problem B is a certain auxiliary problem, equivalent to it, of Riemann–Hilbert type for elliptic systems of three complex equations of first order, whose boundary condition contains no derivatives of the unknown functions.

Theorem 1. Suppose that the function $B(z)$ possesses inside G a generalized derivative $\dot{B} \in L_p(G)$, $p > 2$. If the function $f(z)$ represents a solution of Problem B, then the three functions

$$F_1(z) = f(z), \qquad F_2(z) = \frac{\partial f(z)}{\partial z}, \qquad F_3(z) = \overline{f(z)} \tag{3}$$

satisfy the system of differential equations

$$\frac{\partial F_1}{\partial \bar{z}} = B\bar{F}_1, \qquad \frac{\partial F_2}{\partial \bar{z}} = B_z\bar{F}_1 + |B|^2 F_1, \qquad \frac{\partial F_3}{\partial \bar{z}} = \bar{F}_2 \tag{4}$$

inside the domain G and the boundary conditions

$$\operatorname{Re}[aF_2 + bF_1] = \gamma, \qquad \operatorname{Re}[F_1 - F_3] = 0, \qquad \operatorname{Re}[iF_1 + iF_3] = 0 \tag{5}$$

on the contour Γ. Conversely, if a system of functions F_1, F_2, F_3, continuous in $G + \Gamma$, solves the problem (4), (5), then the first function F_1 gives a solution of the original problem B.

Proof. The direct assertion of the theorem is obtained simply: the first equation (4) is equation (1) in the notation (3). The second equation is obtained from the first by differentiating with the operator $\partial/\partial z$, if after this one again takes into account equation (1). The third equation follows from the second and third relations (3). The first boundary condition (5) is condition (2) in the new notation; the remaining two boundary conditions (5) mean that the real and imaginary parts of the difference $F_1 - \bar{F}_3$, which vanishes identically by virtue of the relations (3), vanish on Γ.

To prove the converse assertion, it is obviously sufficient to establish the identity $F_2(z) \equiv \partial F_1/\partial z$, since then the first boundary condition (5) and the first equation of the system (4) show that the function $F_1(z)$ solves problem B. Differentiating the last equation (4) with the operator $\partial/\partial z$ and taking into account the second equation (4), we obtain

$$\frac{\partial^2 F_3}{\partial z\, \partial \bar{z}} = \frac{\partial \bar{F}_2}{\partial z} = \bar{B}_{zF}1 + |B|^2 \bar{F}_1. \tag{6}$$

In the first identity (4) let us pass to complex conjugate values and differentiate it with the operator $\partial/\partial z$; taking into account the same first identity, we obtain

$$\frac{\partial^2 \bar{F}_1}{\partial z \, \partial \bar{z}} = \frac{\partial}{\partial z}(\bar{B} F_1) = \bar{B}_{zF} 1 + |B|^2 \bar{F}_1. \tag{7}$$

From the identities (6) and (7) it follows that the difference $\bar{F}_1 - F_3$ is a complex-valued solution of the Laplace equation $\Delta(\bar{F}_1 - F_3) = 0$, and since from the last two conditions (5) there follows the identical vanishing of the difference $\bar{F}_1 - F_3$ on Γ, it follows that $\bar{F}_1(z) \equiv F_3(z)$ in the domain G. But then from the last identity (4) we obtain $\partial \bar{F}_3/\partial z = \partial F_1/\partial z = F_2(z)$, which completes the proof of Theorem 1.

2. Let us call the problem (4), (5) problem C. This is a boundary value problem of Riemann—Hilbert type for the elliptic system (4). Every solution of the system (4), continuous in $G + \Gamma$, satisfies the system of integral equations

$$F(z) = -\frac{1}{\pi} \iint_G \frac{A(t)F(t) + B(t)\bar{F}(t)}{t - z} + \Phi(z), \tag{8}$$

where it is denoted that

$$F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 0 & 0 \\ |B|^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} B & 0 & 0 \\ B_z & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \Phi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{F(t)\, dt}{t - z};$$

conversely, any solution of the system (8), where $\Phi(z)$ is an entirely arbitrary vector of analytic functions continuous in $G + \Gamma$, also represents a solution of the system (4). Moreover, the system (8) is solvable for any right-hand side $\Phi(z)$, which readily follows from the solvability of the analogous equation for one function and from the special structure of the matrices A, B. The boundary condition (5), in matrix notation, has the form

$$\mathrm{Re}[g(t)F(t)] = h(t), \qquad g = \begin{pmatrix} b & a & 0 \\ 1 & 0 & -1 \\ i & 0 & i \end{pmatrix}, \qquad h = \begin{pmatrix} \gamma \\ 0 \\ 0 \end{pmatrix}, \tag{9}$$

so that $\det g(t) = -2ia(t)$, $t \in \Gamma$.

We shall assume in what follows that the functions a, b, γ, given on Γ, are Hölder continuous and that $B, B_z \in L_p(G)$, $p > 2$. In addition, we shall assume that $a(t) \neq 0$ for every $t \in \Gamma$, and denote by $\varkappa$ the increment of $\arg a(t)$ when traversing Γ in the positive direction:

$$\varkappa = \frac{1}{2\pi}[\arg\det g(t)]_\Gamma = \frac{1}{2\pi}[\arg a(t)]_\Gamma.$$

Solving system (8) with an arbitrary analytic function $\Phi(z)$ in the right-hand side, we obtain the general representation of all solutions of system (4) that are regular in G and continuous in $G+\Gamma$, in terms of three analytic functions:

$$F(z) = \Phi(z) + \iint_G \left[\Gamma_1(z,t)\Phi(t) + \Gamma_2(z,t)\overline{\Phi(t)}\right]\, d\sigma_t, \tag{10}$$

where $\Gamma_1(z,t)$, $\Gamma_2(z,t)$ are completely determined complex-valued matrix resolvents of system (8). The properties of these resolvent matrices are analogous to the properties of resolvents in the case of a single equation[1].

3. Using representation (10), problem C is reduced to a system of singular integral equations $A\mu = h_1$ with an unknown real vector-function $\mu(t)$[1]; moreover, in view of the assumption $a(t) \neq 0$, $t \in \Gamma$, this system will be of normal type. On the basis of the general theory of systems of singular integral equations of normal type[2], the following theorem is proved:

Theorem 2. The homogeneous problem $\overset{\circ}{C}$ (problem C with $h = 0$), and together with it the homogeneous problem $\overset{\circ}{B}$ (problem B with $\gamma = 0$), have only a finite number of solutions linearly independent over the field of real numbers. For the solvability of problem C, and hence also of problem B, it is necessary and sufficient that the conditions

$$\int_\Gamma h(t)\chi_j(t)\, ds = 0, \qquad j = 1, 2, \dots, q,$$

be satisfied, where χ_j are solutions of the adjoint system of equations $A^*(\chi) = 0$.

4. Let us consider the homogeneous problem $\overset{\circ}{C}_*$, adjoint to problem C.

Problem $\overset{\circ}{C}_*$. It is required to determine a solution, regular in the domain G and continuous in $G+\Gamma$, of the system of differential equations adjoint to (4) (the prime denotes transposition)

$$\frac{\partial \Psi}{\partial z} = -A'\Psi - \overline{B'}\,\overline{\Psi},$$

satisfying on the contour the boundary condition

$$\operatorname{Re}\left[\frac{dt}{ds} g'^{-1}(t)\Psi(t)\right] = 0,$$

where $t = t(s)$ is a point of the contour Γ.

With the aid of Theorem 2, the necessary and sufficient solvability condition for the original problem B is established in terms of solutions of the homogeneous adjoint problem $\overset{\circ}{C}_*$:

Theorem 3. Problem C, and with it also problem B, are solvable if and only if, for every solution $\Psi(z)$ of $\overset{0}{\mathrm{C}}_*$, the conditions

$$\int_\Gamma h(t) g'^{-1}(t) \Psi(t)\, dt = 0;$$

are satisfied; problems C and B are solvable for an arbitrary right-hand side if and only if the problem $\overset{0}{\mathrm{C}}_*$ has no solutions other than the trivial ones.

Let us also denote by l and l_* the numbers of linearly independent solutions of the homogeneous problems $\overset{0}{\mathrm{C}}$ and $\overset{0}{\mathrm{C}}_*$, respectively.

Theorem 4. Between the numbers $\varkappa, l, l_*$, and m there is the relation

$$l - l_* = 2\varkappa - 3(m-1)$$

($(m+1)$ is the connectivity number of the domain G).

From Theorems 3 and 4 it follows, among other things, that if the problem $\overset{0}{\mathrm{B}}$ has no solutions ($l = 0$) and $2\varkappa = 3(m-1)$, then $l_* = 0$; hence problem B is always and uniquely solvable, and conversely (in [3] this result was obtained by another method).

5. In conclusion, we note that we could have assumed the equation to be nonhomogeneous, and also assumed that in (1) and condition (2) there appear not the functions B, a, b, but certain triangular matrices of order n. Then $\varkappa$ will be equal to the increment of $\arg\det a(t)$ along Γ, and the formulations of Theorems 1, 2, and 3 remain unchanged; in Theorem 4, however, we obtain the relation

$$l - l_* = 2\varkappa - 3n(m-1).$$

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Note: Figure translations are in progress. See original paper for figures.