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Abstract

Full Text

THEORY OF ELASTICITY

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ON RELATIONS BETWEEN STRESSES AND STRAINS IN INELASTIC MEDIA

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The problem of relations between stresses and strains in not wholly elastic and plastic media at the present time has grounds for being regarded among those to which thermodynamic methods of investigation are very promising in application. However, in fact with their aid it has so far been possible to construct theories of the behavior of inelastic media only of certain simplest types, namely Voigt elastic-viscous media ⁽¹⁾, some other not wholly elastic media from among those whose basic laws of behavior fit into the general scheme of linear laws of deformation ⁽²⁾, and also the Hencky elastic-plastic medium ^(3,4).

In the present communication we describe an attempt undertaken by us to construct, by thermodynamic means, a “theory of plasticity” in which the generality and other important features of the fundamental laws of thermodynamics would be used to the greatest possible extent.

The basis of the theory, together with the laws of thermodynamics, is formed by the usual general principles of the phenomenological approach, which are, in essence, equivalent to the assumption that it is possible, near each point inside the body under investigation, to single out an element that is a homogeneous thermodynamic system. In accordance with the specific character of the solid state of matter, this system, moreover, must be considered “closed.” As a result, the energy balance for it may be written in the following form:

$$du = \sigma_{ik} d\varepsilon_{ik} + dq, \quad (1)$$

and the second law of thermodynamics in the form of a relation quite equivalent to the classical Clausius relations:

$$dq \leq T ds. \quad (2)$$

Here u and s are the “volume” densities, respectively, of the internal energy and entropy of the element; σ_{ik} are the components, in some system of orthogonal coordinates $X_1 X_2 X_3$, of the stress tensor; $d\varepsilon_{ik}$ is the tensor of a very small change in the strain of the element (reckoned, in the general case, from its

current configuration); dq is the density of the amount of heat received by the element during a very small interval of time; T is its absolute temperature.

As is known, for an ideally elastic continuous deformable medium the complete set of “state arguments” of an element in the general case consists of 7 independent variables, as which, for example, one may take $\sigma_{ik} = \sigma_{ki}$ ($i, k = 1, 2, 3$) and T . In constructing, however, a sufficiently rigorous and general theory of deformable media, reflecting the macroscopic properties and behavior of real solid bodies in deformation processes more fully than the phenomenological theory of elasticity, it must be assumed that the set of “state arguments” of an element, togeth—

along with those just noted, also contains some other parameters. Otherwise, for example, at fixed stresses and temperature the state of the element would always have to remain unchanged, whereas the macroscopic internal state of a real solid under analogous conditions may, as is known, change quite substantially (as a result of “creep” and “relaxation”).

Thus:

$$\begin{aligned} u &= u(\sigma_{11}, \sigma_{12}, \dots, \sigma_{33}, \lambda_1, \lambda_2, \dots, \lambda_r, T), \\ s &= s(\sigma_{11}, \sigma_{12}, \dots, \sigma_{33}, \lambda_1, \lambda_2, \dots, \lambda_r, T), \end{aligned} \quad (3)$$

where $\lambda_1, \lambda_2, \dots, \lambda_r$ denote the indicated “additional” arguments of the state of the element. The introduction of these parameters means, in essence, that, in contrast to the viewpoint usual in the thermodynamics of deformable media, an element of the medium is considered as a heterogeneous homogeneous system, i.e., a system consisting of several homogeneous phases which are, as it were, embedded in one another and are characterized by r independent specific concentrations. In this way, among other things, the only possibility is realized, within the framework of the usual phenomenological approach, of taking into account the presence in real solids of stresses of the second and third kinds, since, evidently, the interaction between the phases mentioned may be interpreted as “hidden” internal forces.

It should be noted that the parameters λ_n , like the stresses σ_{ik} , must be connected with the deformation of the element; moreover, when applied to very small changes in the state of the element, this connection must admit representation by relations of the form

$$d\lambda_n = g_{ik}^{(n)} d\varepsilon_{ik}^p \quad (n = 1, 2, \dots, r), \quad (4)$$

where $g_{ik}^{(n)} = g_{ki}^{(n)}$ are certain coefficients (which may depend on the stresses, deformations, temperature, and, generally speaking, on the history of loading and deformation of the element); $d\varepsilon_{ik}^p$ are quantities determined by the equalities

$$d\varepsilon_{ik}^p = d\varepsilon_{ik} - d\varepsilon_{ik}^e \quad (i, k = 1, 2, 3), \quad (5)$$

in which, in turn, $d\varepsilon_{ik}^e$ are the symbols of quantities representing, if one uses the terminology customary in the mathematical theory of plasticity, the components of the elastic part of the change in deformation of the element. The formal definition of these components in the general case of anisotropy of the thermoelastic properties of the medium is expressed by relations of the form

$$d\varepsilon_{ik}^e = \alpha_{iklm} d\sigma_{lm} + \alpha_{(T)ik} dT \quad (i, k = 1, 2, 3), \quad (6)$$

where $\alpha_{iklm} = \alpha_{kilm} = \alpha_{ikml}$ are the “elasticity coefficients,” and $\alpha_{(T)ik} = \alpha_{(T)ki}$ are the “coefficients of linear thermal expansion” of the material of the medium at the corresponding point.

The noted existence and character of the connections between $d\lambda_n$ and the change in deformation of the element follow from the following considerations. It is known that the set of possible very small changes of any “instantaneous” state of an element of a not entirely elastic medium, whose properties correspond sufficiently well to those of real solids, together with substantially irreversible changes may necessarily contain those for which

$$d\varepsilon_{ik}^p = 0 \quad (i, k = 1, 2, 3). \quad (7)$$

Since the behavior of the element during very small processes of this kind is completely identical to the behavior of an element of some ideally elastic

medium, on a subset of the set of all possible changes of any instantaneous state of the element that constitute these processes, all $d\lambda_n$ must either turn into functions of the changes $d\varepsilon_{ik}$ and dT , or else be identically equal to zero. However, the former, as is not difficult to see, taking into account the physical meaning of the parameters λ_n , would mean that changes of “hidden” stresses under elastic deformations of the element can be comparable with their changes, for example, under cold plastic deformation. Since this plainly contradicts reality, we arrive at the conclusion that the equalities (7) must always have as their consequence the equality to zero of all $d\lambda_n$, whence, in turn, (4) follows.

It is essential that, in the absence in the medium of electromagnetic and thermal fields caused purely by external sources, the equalities (7), roughly speaking, are the necessary and sufficient conditions for the reversibility of a very small change in the thermodynamic state of the element. However, on the basis of the second law, for completely reversible changes of state, the equality sign must hold in relation (2), i.e., the quantity $T ds - dq$ must be equal to zero. It follows from this that this quantity must be representable in the form of an expression analogous in form to the expressions on the right-hand sides of the equalities (4):

$$T ds - dq = \psi_{ik} d\varepsilon_{ik}^p. \quad (8)$$

Since, on the basis of the second law, only such changes of the “instantaneous” states of the element are possible for which relation (2) is valid, this expression must be a positive definite differential form. Therefore the coefficients $\psi_{ik} = \psi_{ki}$ must depend, in particular, on the rates $\eta_{ik}^p = d\varepsilon_{ik}^p/dt$.

In accordance with expression (8), these rates are certain “generalized fluxes,” and the coefficients ψ_{ik} are the corresponding “thermodynamic forces.” In the thermodynamics of irreversible processes, as is known, relations between “fluxes” and “forces” of this kind (“phenomenological relations”) are usually assumed to be linear. On the basis of Onsager’s principle the matrix of the corresponding equations possesses symmetry, which makes it possible to represent the “thermodynamic forces” as partial derivatives, with respect to the corresponding “fluxes,” of some quadratic function of the latter (the “dissipation function”).

However, when applied to systems of the type that interests us, the assumption of linearity of the “phenomenological relations” has considerably fewer grounds than when applied to most systems of the kinds considered in modern thermodynamics of irreversible processes. We shall therefore assume, generalizing Onsager’s principle in a natural way, that

$$\psi_{ik} = \frac{\partial \Psi(\eta_{11}^p, \eta_{12}^p, \dots, \eta_{33}^p, T)}{\partial \eta_{ik}^p} \quad (i, k = 1, 2, 3), \quad (9)$$

where Ψ may be a function of a more complicated type than the usual dissipation function.

From equations (1) and (8) follows the equation

$$du = \sigma_{ik} d\varepsilon_{ik} + T ds - \psi_{ik} d\varepsilon_{ik}^p.$$

Introducing the function $f = f(\sigma_{11}, \sigma_{12}, \dots, \sigma_{33}, \lambda_1, \lambda_2, \dots, \lambda_r, T)$, defined by the equality

$$f = u - Ts, \quad (10)$$

and using (5), (6), and (9), the last equation can be represented in the form

$$\frac{\partial f}{\partial \sigma_{ik}} d\sigma_{ik} + \frac{\partial f}{\partial \lambda_n} g_{ik}^{(n)} d\varepsilon_{ik}^p + \frac{\partial f}{\partial T} dT = \sigma_{lm} \alpha_{lmik} d\sigma_{ik} + \sigma_{ik} d\varepsilon_{ik}^p + \alpha_{(T)lm} \sigma_{lm} dT - s dT - \frac{\partial \Psi}{\partial \eta_{ik}^p} d\varepsilon_{ik}^p. \quad (11)$$

It should be noted that $d\sigma_{ik}$, $d\varepsilon_{ik}^p$, and dT may be regarded as independent quantities. Indeed, let us set $d\varepsilon_{ik}^p = 0$ ($i, k = 1, 2, 3$), i.e., let us consider only completely reversible changes of each prescribed state of the element. Under this condition the following follow from the last equation:

$$\frac{\partial f}{\partial \sigma_{ik}} = \sigma_{lm} \alpha_{lmik} \quad (i, k = 1, 2, 3), \quad (12)$$

$$\frac{\partial f}{\partial T} = -s + \sigma_{lm} \alpha_{(T)lm}. \quad (13)$$

Since these equations relate only the “state arguments,” and moreover in the form of finite dependencies, they must be identities, i.e., must be satisfied for any change of state. From this, as is not difficult to see, the possibility mentioned above follows.

As a result, from equation (11), along with equalities (12) and (13), one obtains the following:

$$\frac{\partial f}{\partial \lambda_n} g_{ik}^{(n)} = \sigma_{ik} - \frac{\partial \Psi}{\partial \eta_{ik}^p} \quad (i, k = 1, 2, 3). \quad (14)$$

The fact that these equations are based, in essence, only on the most general laws and principles of phenomenological physics permits one to hope that they contain the equations of relation between stresses and strains for any deformable continuous medium whose properties are sufficiently close to the corresponding kind of properties of real solids. At the same time, in order to obtain with their help equations describing relations of this kind in media of a quite definite type, it is sufficient to specify the theoretical-analytical properties of the differential forms (4) and the structure of the function Ψ .

Thus, for example, if the forms (4) are completely integrable, and their integrals are functions of the invariants of the tensor $\{\varepsilon_{ik}^p\}$, while Ψ is a function of the invariants of the tensor $\{\eta_{ik}^p\}$, then from equations (14) there ultimately follow the following:

$$\varphi_1 D_\varepsilon^p + \varphi_2 [(D_\varepsilon^p)^2 - \varepsilon_0^p \cdot T_1] = D_\sigma - \omega_1 D_\eta^p - \omega_2 [(D_\eta^p)^2 - \eta_0^p \cdot T_1] \quad (15)$$

(and also the equation relating the “spherical” tensors), where D_ε^p , D_σ , D_η^p denote the deviators, respectively, of the strains ε_{ik}^p , the stresses, and the rates $\dot{\eta}_{ik}^p$; φ_1, φ_2 are functions of the invariants of the tensor $\{\varepsilon_{ik}^p\}$, and ω_1, ω_2 are invariants of the tensor $\{\eta_{ik}^p\}$. These equations describe the relations between stresses and strains in a certain kind of initially isotropic media possessing, along with elasticity, viscosity and “athermal” plasticity.

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