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# MATHEMATICS

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**Abstract**

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## MATHEMATICS

**L. N. IVANOVSKII**

# ON A CONJECTURE OF P. S. ALEXANDROV

*(Presented by Academician P. S. Alexandrov on 21 VII 1958)*

In the present note the following proposition of P. S. Alexandrov is proved:

*Every bicompact topological group is a dyadic bicompactum.*

Let $\theta$ be an arbitrary limit transfinite number. The inverse spectrum $S_\theta = \{G_\alpha, \pi_\beta^\alpha\}$ of bicompact groups over the directed set of all transfinite numbers less than $\theta$ will be called a $\theta$-spectrum if for every limit transfinite number $\alpha < \theta$ the intersection $\bigcap_{\beta<\alpha} \operatorname{Ker} \pi_\beta^\alpha$ contains only the identity element of the group $G_\alpha$. For any limit transfinite number $\alpha < \theta$ the segment $S_{\theta,\alpha}$ of the $\theta$-spectrum $S_\theta$, consisting of the groups $G_\beta$, $\beta < \alpha$, and the homomorphisms $\pi_\beta^\gamma$, $\beta < \gamma < \alpha$, is an $\alpha$-spectrum, whose limit group is, obviously, naturally isomorphic to the group $G_\alpha$. The natural isomorphism $G_\alpha \approx \lim S_{\theta,\alpha}$ will be denoted by $\varphi_\alpha$.

For each $\alpha < \theta$ consider the bicompactum

$$X_\alpha = \prod_{0 \le \mu < \alpha} K_\mu,$$

where $K_\mu$ is a space homeomorphic to the Cantor perfect set (for each $\mu$). It is clear that for any $\beta < \alpha < \theta$ there is defined a natural mapping $\tilde{\pi}_\beta^\alpha : X_\alpha \to X_\beta$. The bicompactum $X_\alpha$ is a zero-dimensional topological group, and the mappings $\tilde{\pi}_\beta^\alpha$ are continuous homomorphisms. The groups $X_\alpha$, $\alpha < \theta$, and the homomorphisms $\tilde{\pi}_\beta^\alpha$, $\beta < \alpha < \theta$, form, obviously, a $\theta$-spectrum $K(\theta)$. The limit group of this spectrum is the group $D_\tau$, where $\tau$ is the cardinality of the number $\theta$.

Following L. S. Pontryagin ([2]), we shall call a $\theta$-spectrum $S_\theta$ a Li row of length $\theta$ if the group $G_1$ is a Li group, and for every transfinite number $\alpha < \theta$ the homomorphism

$$\pi_\alpha^{\alpha+1} : G_{\alpha+1} \to G_\alpha$$

is an epimorphism whose kernel is some Li group.

A system of continuous (generally speaking, nonhomomorphic!) mappings

$$f^\beta : X_\beta \to G_\beta, \quad \beta < \alpha \le \theta,$$

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will be called an $\alpha$-special mapping of the $\theta$-spectrum $K(\theta)$ into the Li row $S_\theta$ if: a) the image of the mapping $f^1 : X_1 \to G_1$ coincides with the whole group $G_1$; b)

$$\pi_\beta^\gamma f^\gamma = f^\beta \tilde{\pi}_\beta^\gamma, \quad \beta < \gamma < \alpha;$$

c) if

$$f^\beta(z) = \pi_\beta^{\beta+1}(y) = x,$$

where $x \in G_\beta$, $y \in G_{\beta+1}$, $z \in X_\beta$, $\beta + 1 < \alpha$, then there exists such an element $\vartheta \in X_{\beta+1}$ that

$$\tilde{\pi}_\beta^{\beta+1}(\vartheta) = z$$

and

$$f^{\beta+1}(\vartheta) = y.$$

A $\beta$-special mapping $g : K(\theta) \to S_\theta$ is called an extension of an $\alpha$-special mapping $f : K(\theta) \to S_\theta$, $\alpha < \beta$, if $g^\delta = f^\delta$ for $\delta < \alpha$.

**Lemma 1.** *For any $\theta$-special mapping $f : K(\theta) \to S_\theta$, the image of the limit mapping*

$$f^* : \lim K(\theta) \to \lim S_\theta$$

*coincides with the whole group* $\lim S_\theta$.

For the proof, consider an arbitrary thread $\{x_\alpha\}$ of the Li row $S_\theta$ and suppose that for each $\alpha'$, less than some $\alpha < \theta$, such elements

$$y_1, y_2, \dots, y_{\alpha'}, \dots, \ y_{\alpha'} \in X_{\alpha'}$$

have already been defined that

$$f^{\alpha'}(y_{\alpha'}) = x_{\alpha'}.$$

$\alpha' < \alpha$, $\pi_{\alpha''}^{\alpha'}(y_{\alpha'}) = y_{\alpha''}$, $\alpha'' < \alpha' < \alpha$, $y_1 \in (f^1)^{-1}(x_1)$. If $\alpha$ is a limit transfinite number, then the elements $y_1, y_2, \dots, y_{\alpha'}, \dots,\ \alpha' < \alpha$, form a thread $\omega$ of the $\alpha$-spectrum $K(\theta)_\alpha$, and therefore the element $y_\alpha = \tilde{\varphi}_\alpha^{-1}(\omega) \in X_\alpha$ is defined. If $\alpha$ is a non-limit transfinite number, then by property c) there is an element $y_\alpha \in X_\alpha$ such that $\pi_{\alpha-1}^\alpha(y_\alpha) = y_{\alpha-1}$ and $f^\alpha(y_\alpha) = x_\alpha$. Continuing the process, we obtain a thread $\{y_\beta\}$ of the $\theta$-spectrum $K(\theta)$, which is carried under the mapping $f^* : \lim K(\theta) \to \lim S_\theta$ into the thread $\{x_\beta\}$ of the Li series $S_\theta$.

**Lemma 2.** *Let $f : G \to H$ be a homomorphic mapping of the group $G$ onto the group $H$, whose kernel* $\operatorname{Ker} f$ *is a compact Lie group, and let $g : N \to H$ be a continuous mapping of a zero-dimensional bicompactum $N$ into the group $H$. Then there exists a continuous mapping $g' : N \to G$ such that $fg' = g$.*

For the proof, note that $(G, H, f, \operatorname{Ker} f)$ is, by Gleason' s theorem ([3]), a fiber bundle, and therefore there exists an open covering $\{u_i\}$ of the group $H$ and continuous mappings $\varphi_i : u_i \to G$ such that $f\varphi_i = 1_{u_i}$. Into the open covering $\{g^{-1}(u_i)\}$ of the zero-dimensional bicompactum $N$ we now inscribe a finite open

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covering $\{V_1, \ldots, V_r\}$ consisting of pairwise disjoint sets, and define the mapping $g' : N \to G$ by setting $g'(x) = \varphi_i g(x)$ for all $x \in V_j \subset g^{-1}(u_i)$.

**Lemma 3.** *For every $\alpha$-special ($\alpha < \theta$) mapping $f : K(\theta) \to S_\theta$ of the $\theta$-spectrum $K(\theta)$ into the Li series $S_\theta$, there exists an $(\alpha+1)$-special mapping $g : K(\theta) \to S_\theta$ extending the mapping $f$.*

If $\alpha$ is a limit transfinite number, then, setting $f^\alpha = \tilde{\varphi}_\alpha^{-1} f^* \varphi_\alpha$, where $f^* : \lim K(\theta)_\alpha \to \lim S_{\theta,\alpha}$ is the mapping induced by the mapping $f : K(\theta) \to S_\theta$, we obtain a certain mapping $f^\alpha : X_\alpha \to G_\alpha$. If $\alpha$ is a non-limit transfinite number, then, using Lemma 2, we can construct a continuous mapping $f' : X_{\alpha-1} \to G_\alpha$ such that $\pi^\alpha_{\alpha-1} f' = f^{\alpha-1}$. Consider, in addition, the natural isomorphism $q : X_\alpha \approx X_{\alpha-1} \times K_{\alpha-1}$ and some continuous mapping $p : K_{\alpha-1} \to \operatorname{Ker} \pi^\alpha_{\alpha-1}$ of the Cantor perfect set $K_{\alpha-1}$ onto the compact Lie group $\operatorname{Ker} \pi^\alpha_{\alpha-1}$ (see (1)). We now define the mapping $q' : X_{\alpha-1} \times K_{\alpha-1} \to G_\alpha$ by the formula $q'(x, a) = f'(x) \cdot p(a)$, where $x \in X_{\alpha-1}$, $a \in K_{\alpha-1}$, and put $f^\alpha = q' q : X_\alpha \to G_\alpha$. It is easy to see that the mapping $f^\alpha$, together with the mappings $f^\beta$, where $\beta < \alpha$, defines an $(\alpha+1)$-special mapping $g : K(\theta) \to S_\theta$ extending the mapping $f$.

**Corollary.** *There exists a $\theta$-special mapping $f : K(\theta) \to S_\theta$ of the $\theta$-spectrum $K(\theta)$ into the Li series $S_\theta$.*

**Theorem.** *For every bicompact group $G$ of weight $\tau$ there exists a continuous mapping $f : D_\tau \to G$ of the group $D_\tau$ onto the whole group $G$.*

For the proof it suffices to note that, by a theorem of L. S. Pontryagin ([2]), there exists a Li series $S_\theta$ of length $\theta$, where $\theta$ is the first transfinite number of cardinality $\tau$, having as its limit group the group $G$.

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## REFERENCES

[1] P. S. Aleksandrov, *Introduction to the General Theory of Sets and Functions*, Moscow-Leningrad, 1948. [2] L. S. Pontryagin, *Continuous Groups*, Moscow, 1954. [3] A. M. Gleason, Proc. Am. Math. Soc., 1, 35 (1950).

*Note: Figure translations are in progress. See original paper for figures.*

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