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Abstract

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A. A. Govorukhina

Integro-Differential Equations of Convolution Type

(Presented by Academician V. I. Smirnov on 29 VIII 1957)

In this paper we consider integro-differential equations

$$\sum_{m=0}^{n}\left[\lambda_m f^{(m)}(x)+\frac{1}{\sqrt{2\pi}}\int_0^{\infty}k_{1m}(x-t)f^{(m)}(t)\,dt+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}k_{2m}(x-t)f^{(m)}(t)\,dt\right]=g(x) \quad \text{(A)}$$

and paired equations

$$\sum_{m=0}^{n}\left[\lambda_m f^{(m)}(x)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}k_{1m}(x-t)f^{(m)}(t)\,dt\right]=g_1(x), \qquad x>0;$$

$$\sum_{m=0}^{n}\left[\mu_m f^{(m)}(x)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}k_{2m}(x-t)f^{(m)}(t)\,dt\right]=g_2(x), \qquad x<0. \quad \text{(B)}$$

The kernels and the free term are taken from the classes*

$$k_{im+}(x)e^{-yx}\in L(-\infty,\infty) \quad \text{for } y\geq a_{im},$$

$$k_{im-}(x)e^{-yx}\in L(-\infty,\infty) \quad \text{for } y\leq b_{im}, \qquad i=1,2; \quad (1)$$

$$g_+(x)e^{-yx}\in L^p(-\infty,\infty) \quad y\geq a,$$

$$g_-(x)e^{-yx}\in L^p(-\infty,\infty) \quad \text{for } y\leq b. \quad (2)$$

The solution is sought in the class

$$f_+^{(m)}(x)e^{-yx} \in L^p(-\infty, \infty) \quad \text{for } y \geq \alpha,$$

$$f_-^{(m)}(x)e^{-yx} \in L^p(-\infty, \infty) \quad \text{for } y \leq \beta \tag{3}$$

for all m and $1 < p \leq 2$.

We shall denote such classes by the symbols

$$f_+(x) \in \{\alpha, \infty\}_p; \qquad f_-(x) \in \{-\infty, \beta\}_p \quad \text{or} \quad f(x) \in \{\alpha, \beta\}_p.$$

The simplest integro-differential equations of the type under consideration were solved for the first time by I. M. Rapoport ([1]).

Using the theory of Fourier transforms and the investigations of F. D. Gakhov

* Functions identically equal to zero for $x < 0$ or $x > 0$ will be denoted, respectively, by the signs $+$ or $-$:

$$f_+(x) \equiv 0 \quad \text{for } x < 0; \qquad f_-(x) \equiv 0 \quad \text{for } x > 0.$$

and Yu. N. Cherskii ([2,3]) in the field of integral equations of convolution type, we have obtained the following results.

Equations (A)

Theorem 1. If $k_{im}(x) \in L(-\infty, \infty)$, $i = 1, 2$, $g(x) \in L^p(-\infty, \infty)$, and the solution u is sought in the class $f^{(m)}(x) \in L^p(-\infty, \infty)$, $m = 0, 1, \dots, n$, then equation (A) is equivalent to the Riemann boundary-value problem

$$\Phi_n^+(x) = A(x)\Phi_n^-(x) + B(x), \qquad -\infty < x < \infty, \tag{4}$$

with the additional conditions

$$\left.\frac{d^m \Phi_n^{\pm}(z)}{dz^m}\right|_{z=0} + \frac{m!(-i)^m}{\sqrt{2\pi}} f^{(n-m-1)}(0) = 0, \qquad m = 0, 1, \dots, n-1, \tag{5}$$

where $A(x), B(x)$ are functions expressed in a definite way in terms of $K_{im}(x), G(x)$—the Fourier transforms of the kernels and of the right-hand side of equation (A); $\Phi_n^{\pm}(z)$ are the Fourier transforms of the functions $f_{\pm}^{(n)}(x)$; $\Phi_n^+(z), \Phi_n^-(z)$ are functions analytic, respectively, in the upper and lower half-planes.

Theorem 2. Under assumptions (1), (2), (3), the widest admissible class for the right-hand side is

$$g_+(x) \in \left\{ \max\left[\max_{m=0,1,\dots,n} (a_{1m}, \min_{k=0,1,\dots,n} b_{1k}), \max_{k=0,1,\dots,n} b_{2k} \right], \infty \right\}_p,$$

$$g_-(x) \in \left\{ -\infty, \min\left[\min_{m=0,1,\dots,n} (b_{2m}, \max_{m=0,1,\dots,n} a_{2k}), \min_{k=0,1,\dots,n} b_{1k} \right] \right\}_p, \qquad (6)$$

and the widest class of functions admissible as solutions of equation (A) is

$$f^{(m)}(x) \in \left\{ \min_{m=0,1,\dots,n} b_{1m}, \max_{k=0,1,\dots,n} a_{2k} \right\}_p, \qquad m = 0, 1, \dots, n. \qquad (7)$$

Remark. The numbers a_{1k} and b_{2k}, $k = 0, 1, \dots, n$, have no influence on the choice of the solution class.

Definition 1. If the integro-differential equation (A) can be represented in the form of a sum of terms $\varphi_i(x)$ such that $\varphi_i(x) \in \{\gamma_i, \delta_i\}_p$, where $[\gamma_i, \delta_i]$ are pairwise nonintersecting segments, then the number of such terms will be called the **rank of the equation**.

Theorem 3. The rank of equation (A) is not higher than 4.

Depending on the choice of relations between the numbers a_{ik}, b_{jm}, $i = 1, 2$, $j = 1, 2$, m and $k = 0, 1, \dots, n$, the rank of the equation changes, and after the Fourier transform various boundary-value problems are obtained. The numbers a_{ik}, b_{jm} can be arranged in $(4n + 4)!$ ways. Therefore, theoretically the same number of different boundary-value problems is conceivable. All problems can be subdivided into two essentially different types.

Definition 2. Integro-differential equations reducible to the solution only of Riemann boundary-value problems will be called **equations of the type of Riemann boundary-value problems**. The investigation shows that this group of equations is characterized by the conditions

$$b_1 = \min_{m=0,1,\dots,n} b_{1m} \geq \max_{k=0,\dots,n} a_{2k} = a_2. \qquad (8)$$

Definition 3. Integro-differential equations reducible to "strip problems" ([4]) will be called **equations of strip type**. For them the defining inequality is

$$b_1 < a_2. \qquad (9)$$

Theorem 4. If $K^+_{1\xi}(z)$ for all m and $G^+(z)$ are analytically continuable with admissible poles at a finite number of points to the straight line:

$x + ib_1$, and $K^-_{2m}(z)$ for all m and $G^-(z)$ are analytically continuable, with admissible poles at a finite number of points, up to the straight line $x+ia_2$, and the functions

$$\frac{1}{\sqrt{2\pi}} \sum_{m=0}^{n-1} (-iz)^{m-n} K^+_{1m}(z) \sum_{q=0}^{n-m-1} (-iz)^q f^{(n-q-1)}(0) - G^+(z) = P_1(z), \qquad (10)$$

$$\frac{1}{\sqrt{2\pi}} \sum_{m=0}^{n-1} (-iz)^{m-n} K^-_{2m}(z) \sum_{q=0}^{n-m-1} (-iz)^q f^{(n-q-1)}(0) + G^-(z) = P_2(z) \qquad (11)$$

have in the continued strip poles only where the corresponding functions

$$K^+_1(z) = \sum_{m=0}^{n} (-iz)^{m-n} K^+_{1m}(z), \qquad (12)$$

$$K^-_2(z) = \sum_{m=0}^{n} (-iz)^{m-n} K^-_{2m}(z), \qquad (13)$$

have them, and of no higher orders, then equation (A) of the Riemann boundary-value-problem type is equivalent* to the Riemann boundary-value problem itself on the contour γ, consisting of two parallel straight lines $x + ia_2$ and $x + ib_1$:

$$\psi^+(\zeta) = A(\zeta)\psi^-(\zeta) + B(\zeta), \qquad \zeta \in \gamma, \qquad (14)$$

where

$$\psi(z) = \begin{cases} \Phi^+_n(z), & \text{for } \operatorname{Im} z > b_1, \\ \Omega(z), & \text{for } a_2 < \operatorname{Im} z < b_1, \\ \Phi^-_n(z), & \text{for } \operatorname{Im} z < a_2; \end{cases} \qquad (15)$$

$\Omega(z)$ is a certain auxiliary function analytic in the strip $a_2 < \operatorname{Im} z < b_1$; $A(z)$, $B(z)$ are functions expressible in a definite way through $K_{1m}(z)$, $K_{2m}(z)$, $G(z)$, and the solution must satisfy the additional conditions:

1) $$\left.\frac{d^j \Phi^+_n(z)}{dz^j}\right|_{z=z_k} = -\left.\frac{d^j}{dz^j}\left[\frac{P_1(z)}{K^+_1(z)}\right]\right|_{z=z_k}; \qquad \left.\frac{d^l \Phi^-_n(z)}{dz^l}\right|_{z=z_t} = -\left.\frac{d^l}{dz^l}\left[\frac{P_2(z)}{K^-_2(z)}\right]\right|_{z=z_t};$$

$$j = 0, 1, \ldots, \nu_k - 1; \qquad l = 0, 1, \ldots, \nu_t - 1, \qquad (16)$$

z_k is a pole of $K_1^+(z)$ of multiplicity ν_k; z_t is a pole of $K_2^-(z)$ of multiplicity ν_t;

$$2)\quad \int_{-\infty}^{\infty} |\Phi_m^{\pm}(x+iy)|^{p'}\,dx, \qquad m=0,1,\dots,n \quad \left(\frac{1}{p}+\frac{1}{p'}=1\right) \tag{17}$$

are uniformly bounded for $y \geq b_1$ $(y \leq a_2)$;

$$\int_{-\infty}^{\infty} |\Omega(x+iy)|^{p'}\,dx$$

is uniformly bounded for $a_2 < y < b_1$.

Consequence. Under the assumptions of Theorem 4, equations (A) of the type of Riemann boundary-value problems are solved in closed form.

Remark 1. Conditions (16) and (17) ensure the representability of $\Phi_m^{\pm}(z)$ and $\Omega(z)$ in the form of Fourier integrals.

* Equivalence is meant in the usual sense.

Remark 2. In some cases the conditions of the theorem are satisfied automatically. For example, for $a_1 \leqslant b_1$ and $b_2 \geqslant a_2$, $G^+(z)$, $K_{1m}^+(z)$ are analytic functions up to the line $x+ib_1$, while $G^-(z)$ and $K_{2m}^-(z)$ are analytic functions up to the line $x+ia_2$.

Theorem 5. If $K_{1m}^+(z)$, $K_{2m}^+(z)$, and $G^+(z)$ are analytically continuable with poles to the line $x+ib_1$, while $K_{1m}^-(z)$, $K_{2m}^-(z)$, and $G^-(z)$ are analytically continuable with poles to the line $x+ia_2$, and the functions

$$P_1(z); \qquad P_1(z)+\frac{1}{\sqrt{2\pi}}\sum_{m=0}^{n-1}(-iz)^{m-n}\left[\lambda_m-K_{2m}^+(z)\right]\sum_{q=0}^{n-m-1}(-iz)^q f^{(n-q-1)}(0);$$

$$P_2(z); \qquad P_2(z)+\frac{1}{\sqrt{2\pi}}\sum_{m=0}^{n-1}(-iz)^{m-n}\left[\lambda_m-K_{1m}^-(z)\right]\sum_{q=0}^{n-m-1}(-iz)^q f^{(n-q-1)}(0)$$

have, in the continued strips, poles only where they are present, respectively, in the functions

$$K_1^+(z); \qquad \sum_{m=0}^{n}(-iz)^{m-n}[\lambda_m+K_{2m}^+(z)];$$

$$\sum_{m=0}^{n}(-iz)^{m-n}[\lambda_m-K_{2m}^-(z)]; \qquad K_2^-(z)$$

and of no higher orders, then the equation (A) of "area" type is equivalent (see the remark to Theorem 4) to the Riemann boundary-value problem on the contour γ, consisting of two parallel lines $x + ib_1$ and $x + ia_2$:

$$\psi^+(\zeta) = A(\zeta)\psi^-(\zeta) + B(\zeta), \qquad \zeta \in \gamma,$$

with the additional conditions (15), (16), (17) and the conditions ensuring analyticity of the functions $\Phi_n^\pm(z)$ in the strip $b_1 < \operatorname{Im} z < a_2$ (the general form of which would be difficult to write down).

Corollary. Under the assumptions of Theorem 5, the equations (A) of "area" type are solved in finite form.

Analogous theorems hold for equations ( ).

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Rostov-on-Don
State University

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Note: Figure translations are in progress. See original paper for figures.