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## Abstract

## Full Text

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## MATHEMATICS

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# FOURIER SERIES AS A DEVICE FOR APPROXIMATING ALMOST-PERIODIC FUNCTIONS

*(Presented by Academician V. I. Smirnov on 30 V 1958)*

1. One of the questions in the theory of harmonic approximation is that of establishing the dependence between the deviations of the partial sums of the Fourier series of a continuous  $2\pi$ -periodic function and the best approximations of this function by trigonometric polynomials.

In the present note we briefly set forth some results of considering an analogous question for almost-periodic functions. Namely, a dependence is established between the deviations of the partial sums of the Fourier series of an almost-periodic function, whose Fourier exponents have no finite limit points, and the best approximations of this function by entire functions of finite degree.

Let the Fourier series of the almost-periodic function  $f(x)$  be written in the following form:

$$f(x) \sim \sum_{k=-\infty}^{\infty} A_k e^{i\Lambda_k x} \quad (1)$$

$$(\Lambda_0 = 0; \quad \Lambda_k < \Lambda_{k+1} \text{ for } k = 0, 1, 2, \dots;$$

$$\lim_{k \rightarrow \infty} \Lambda_k = \infty, \quad \Lambda_k = -\Lambda_{-k}; \quad |A_k| + |A_{-k}| \neq 0 \text{ for } k \neq 0).$$

Denote by  $L = L(f)$  the sequence  $\{\Lambda_k\}$  ( $k = 1, 2, \dots$ ). Put

$$R_\lambda(f) = \sup_x \left| f(x) - \sum_{|\Lambda_k| \leq \lambda} A_k e^{i\Lambda_k x} \right|;$$

$$\alpha_\lambda(f) = \sum_{|\Lambda_k| > \lambda} |A_k|, \quad E_\lambda(f) = \inf_{F(z) \in B_\lambda} \left\{ \sup_x |f(x) - F(x)| \right\},$$

where  $B_\lambda$  is the class of entire functions of degree  $\leq \lambda$ , bounded on the real axis.

Let  $l = \{\lambda_k\}$  ( $k = 1, 2, \dots$ ;  $0 < \lambda_k < \lambda_{k+1}$ ) be an increasing sequence of positive numbers. Put

$$N_l(\lambda) = \sum_{\lambda_k \leq \lambda} 1.$$

We shall say that the sequence  $l$  belongs to the class  $A$  if there is an  $a > 0$  such that  $\lambda_{k+1} - \lambda_k > a$  ( $k = 1, 2, \dots$ ). Denote by  $A_\sigma$  the class of all sequences  $l$ , each of which can be partitioned into a finite number of sequences belonging to the class  $A$ . Thus,  $l \in A_\sigma$  if the representation

$$l = \bigcup_{j=1}^r l^{(j)}, \quad \text{where } l^{(j)} \in A \quad (j = 1, 2, \dots, r)$$

is possible.

Let  $\mathcal{L}$  be the class of all lacunary sequences  $l$ . A sequence  $l$  belongs to the class  $\mathcal{L}$  if there exists  $\theta > 1$  such that

$\lambda_{k+1}/\lambda_k \geq \theta$ . By  $\mathcal{L}_\sigma$  we denote the class of all sequences  $l$ , each of which can be divided into a finite number of lacunary ones. Thus,  $\mathcal{L}_\sigma$  is the class of sequences  $l$  admitting the representation

$$l = \bigcup_{j=1}^r l^{(j)}, \quad \text{where } l^{(j)} \in \mathcal{L} \quad (j = 1, 2, \dots, r).$$

The inclusions  $A_\sigma \supset A$ ,  $\mathcal{L}_\sigma \supset \mathcal{L}$  are obvious.

2. We formulate the main theorem of the note.

**Theorem 1.** *Let  $0 < \lambda < \mu$ ; then, for any almost-periodic function  $f(x)$  whose Fourier exponents have no finite limit points, the inequality*

$$R_\lambda(f) \leq \Phi(\lambda, \mu) E_\lambda(f), \tag{2}$$

holds, where

$$\Phi(\lambda, \mu) = 1 + \frac{4}{\pi} + 2[N_L(\mu) - N_L(\lambda)] + \frac{2}{\pi} \ln \frac{\mu + \lambda}{\mu - \lambda}.$$

The method of proof of Theorem 1 is a generalization and development of the method set out in the author's paper <sup>(7)</sup>.

In applications of Theorem 1 the parameter  $\mu$  is chosen so as to minimize, as far as possible, the factor  $\Phi(\lambda, \mu)$ .

From Theorem 1, formulated under very general conditions, a number of theorems follow. Let us consider some of them.

3. Let  $f(x)$  be a continuous  $2\pi$ -periodic function, and let  $E_n^*(f)$  be the best approximation of this function by trigonometric polynomials of order  $n$ . Then (see <sup>(1)</sup>, pp. 374-375)  $E_n^*(f) = E_n(f)$ . Putting in inequality (2)  $\lambda = n$ ,  $\mu = n + 1 - \varepsilon$ , where  $\varepsilon$  is sufficiently small, we obtain Lebesgue's theorem <sup>(2)</sup>, pp. 193-194).

The following theorem is a generalization of Lebesgue's theorem to the almost-periodic case.

**Theorem 2.** *Let the sequence  $L(f)$  have the following property: there exists a function  $\varphi(\lambda)$ , nonnegative for  $\lambda > \lambda_0$ , such that*

$$N_L\left(\lambda + \frac{\lambda}{e^{\varphi(\lambda)}}\right) - N_L(\lambda) = O[1 + \varphi(\lambda)].$$

*Then  $R_\lambda(f) \leq \Phi(\lambda)E_\lambda(f)$ , where  $\Phi(\lambda) = O[1 + \varphi(\lambda)]$ .*

For the proof of Theorem 2 it is enough to put in inequality (2)

$$\mu = \lambda + \frac{\lambda}{e^{\varphi(\lambda)}}.$$

We note two theorems contained in Theorem 2.

**Theorem 3.** *If  $L(f) \in A_\sigma$ , then  $R_\lambda(f) \leq \Phi(\lambda)E_\lambda(f)$ , where  $\Phi(\lambda) = O(\ln \lambda)$ .*

**Proof.** For  $L(f) \in A_\sigma$  there exists  $a > 0$  such that

$$N_L(\lambda + a) - N_L(\lambda) = O(1).$$

Putting in Theorem 2  $\varphi(\lambda) = \ln \frac{\lambda}{a}$ , we obtain the assertion to be proved.

**Theorem 4.** *If  $L(f) \in \mathcal{L}_\sigma$ , then  $R_\lambda(f) = O[E_\lambda(f)]$ .*

**Proof.** It is known <sup>(4)</sup> that for  $L(f) \in \mathcal{L}_\sigma$  we have  $N_L(2\lambda) - N_L(\lambda) = O(1)$ ; therefore the condition of Theorem 2 is fulfilled for  $\varphi(\lambda) = 0$ .

This theorem is a generalization, to almost-periodic functions, of Theorem 4 of the paper <sup>(4)</sup>.

4. Let us consider theorems providing new criteria for the uniform and absolute convergence of Fourier series of almost-periodic functions. Here the following theorem is the starting point.

**Theorem 5.** *The Fourier series (1) of an almost-periodic function  $f(x)$  converges uniformly if there exists a numerical sequence  $\{\mu_n\}$  ( $n = 1, 2, \dots$ ) satisfying the conditions:*

- 1)  $\mu_n > \Lambda_n$  for  $n > n_0$ ;
- 2)  $\lim_{n \rightarrow \infty} E_{\Lambda_n}(f) [N_L(\mu_n) - N_L(\Lambda_n)] = 0$ ;
- 3)  $\lim_{n \rightarrow \infty} E_{\Lambda_n}(f) \ln \frac{\mu_n + \Lambda_n}{\mu_n - \Lambda_n} = 0$ .

To prove Theorem 5 it is enough to put  $\mu = \mu_n$ ,  $\lambda = \Lambda_n$  in inequality (2) and take into account that  $\lim_{\lambda \rightarrow \infty} E_\lambda(f) = 0$  ((<sup>1</sup>), pp. 371-372).

**Theorem 6.** The Fourier series (1) of the almost-periodic function  $f(x)$  converges uniformly if

$$\lim_{n \rightarrow \infty} E_{\Lambda_n}(f) \ln \frac{\Lambda_{n+1} + \Lambda_n}{\Lambda_{n+1} - \Lambda_n} = 0.$$

Theorem 6 follows from Theorem 5 for  $\mu_n = \Lambda_{n+1}$ .

Let us note a consequence of Theorem 6, which holds by virtue of the inequality

$$E_\lambda(f) \leq 2\pi\omega_f\left(\frac{1}{\lambda}\right), \quad \text{where } \omega_f(\delta) = \sup_{|x-y| \leq \delta} |f(x) - f(y)|$$

((<sup>1</sup>), pp. 371-373).

**Corollary.** The Fourier series (1) of the almost-periodic function  $f(x)$  converges uniformly if

$$\lim_{n \rightarrow \infty} \omega_f\left(\frac{1}{\Lambda_n}\right) \ln \frac{\Lambda_{n+1} + \Lambda_n}{\Lambda_{n+1} - \Lambda_n} = 0.$$

Theorem 6, which is a generalization of Bohr's criterion ((<sup>3</sup>), pp. 81-83), makes it possible to use more fully the properties of the modulus of continuity of an almost-periodic function when investigating its Fourier series for uniform convergence.

It is easy to see that the condition of Theorem 6 ensuring uniform convergence, when the sequence  $L(f)$  of frequencies contains gaps whose lengths decrease arbitrarily rapidly, can be fulfilled only at the expense of a correspondingly rapid decrease of the best approximation of the almost-periodic function.

From the theorems given below it follows that this undesirable circumstance is caused not by the essence of the matter, but by the choice of the sequence  $\{\mu_n\}$  in Theorem 6.

**Theorem 7.** The Fourier series (1) of the almost-periodic function  $f(x)$  converges uniformly if there exists a function  $\varphi(\lambda) \geq 0$  for  $\lambda > \lambda_0$ , satisfying the conditions:

- 1)  $N_L\left(\Lambda_n + \frac{\Lambda_n}{e^{\varphi(\Lambda_n)}}\right) - N_L(\Lambda_n) = O[1 + \varphi(\Lambda_n)];$
- 2)  $\lim_{n \rightarrow \infty} \varphi(\Lambda_n) E_{\Lambda_n}(f) = 0.$

Theorem 7 follows from Theorem 5 for

$$\mu_n = \Lambda_n + \frac{\Lambda_n}{e^{\varphi(\Lambda_n)}}.$$

**Theorem 8.** Suppose that the sequence  $L(f)$  has the following property: there are numbers  $a > 0$ ,  $m \geq 0$  such that

$$N_L\left(\Lambda_n + \frac{a}{\Lambda_n^m}\right) - N_L(\Lambda_n) = O(\ln \Lambda_n).$$

Then the Fourier series (1) of the almost-periodic function  $f(x)$  converges uniformly if  $\lim_{\delta \rightarrow 0} \omega_f(\delta) \ln \delta = 0$ .

Theorem 8 follows from Theorem 7 for  $\varphi(\lambda) = \ln \frac{\lambda^{m+1}}{a}$ .

**Corollary.** If  $L(f) \in A_\sigma$  and  $\lim_{\delta \rightarrow 0} \omega_f(\delta) \ln \delta = 0$ , then the Fourier series (1) of the almost-periodic function  $f(x)$  converges uniformly.

Theorem 8 is a generalization to the almost-periodic case of the Dini-Lipschitz criterion <sup>(5,2)</sup>.

**Theorem 9.** If  $L(f) \in \mathcal{L}_\sigma$ , then the Fourier series (1) of the almost-periodic function  $f(x)$  converges absolutely; moreover, if  $A_0 = 0$ , then

$$\sum_{k=-\infty}^{\infty} |A_k| \leq C_L \sup_x |f(x)|,$$

where  $C_L$  is a constant depending only on the sequence  $L(f)$ .

The proof of Theorem 9 is analogous to the proof of Theorem 1 of the author's note <sup>(6)</sup>. In the proof of Theorem 9, Theorem 1 of the paper <sup>(4)</sup> and the uniform convergence of the series (1), following from Theorem 7 for  $\varphi(\lambda) = 0$ , are used in an essential way.

5. In conclusion we formulate a theorem generalizing the results of the author's paper <sup>(7)</sup> concerning best approximations of almost-periodic functions representable by lacunary series.

**Theorem 10.** If  $L(f) \in \mathcal{L}_\sigma$ , then the order equalities

$$E_\lambda(f) \sim R_\lambda(f) \sim \alpha_\lambda(f)$$

hold.

The proof of Theorem 10 is based on the application of Theorems 4 and 9 of the present note.

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*Note: Figure translations are in progress. See original paper for figures.*

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