Mathematics

1958

SovietRxiv

Abstract

Full Text

Mathematics

I. Kh. Khairullin

On Certain Infinite Systems of Linear Algebraic Equations Solvable in Closed Form

(Presented by Academician P. Ya. Kochina, 15 VII 1958)

In the present note systems of the following types are investigated:

$$x_n + \sum_{k=-\infty}^{\infty} a_{n-k} x_k = d_n \qquad (n < 0),$$

$$x_n + \sum_{k=-\infty}^{\infty} b_{n,k} x_k = d_n \qquad (0 \le n \le p-1), \tag{1}$$

$$x_n + \sum_{k=-\infty}^{\infty} c_{n-k} x_k = d_n \qquad (p \le n);$$

$$x_n + \sum_{k=-\infty}^{\infty} \left[1 + e^{2k\pi i/m} + \cdots + e^{2(m-1)k\pi i/m}\right] a_{n-k} x_k = d_n \qquad (n < 0),$$

$$x_n + \sum_{k=-\infty}^{\infty} \left[1 + e^{2k\pi i/m} + \cdots + e^{2(m-1)k\pi i/m}\right] c_{n-k} x_k = d_n \qquad (0 \le n), \tag{2}$$

where m is a positive integer.

A certain analogy of these systems with convolution-type integral equations ($^{1-7}$) and singular integral equations whose kernels remain invariant under substitutions of a certain group of fractional-linear transformations (^{8}) is used. As the apparatus for investigating the indicated systems, Laurent transformations* (^{9}) and the theory of boundary-value problems for analytic functions (^{10}) are applied.

One class of infinite systems of linear algebraic equations with difference indices was considered in (^{9}, ^{11}). J. N. Feld considers infinite systems connected with problems on semi-infinite periodic structures. We shall use the terminology of (^{9}).

For Laurent transformations the following theorem on convolutions holds.

Theorem. If the sequence $\{a_n\rho^n\} \in l_1$ for any ρ from the interval $[\alpha_1, \beta_1]$, and $\{x_n\rho^n\} \in l_2$ for any ρ from the interval $[\alpha_2, \beta_2]$, and

$$\max(\alpha_1, \alpha_2) \leq \min(\beta_1, \beta_2),$$

then:

1) $\left\{\rho^n \sum_{k=-\infty}^{\infty} a_{n-k}x_k\right\} \in l_2$ for any ρ from the interval $[\max(\alpha_1, \alpha_2), \min(\beta_1, \beta_2)]$;

2) $A(z)X(z)$ will be the image of $\left\{\sum_{k=-\infty}^{\infty} a_{n-k}x_k\right\}$.

We do not give the proofs.

§ 1. Consider the infinite system (1).

Case 1. Let

$$|a_n| < \frac{M}{|n|^{\lambda+1}}, \qquad |c_n| < \frac{M}{|n|^{\lambda+1}} \quad (0 < \lambda \leq 1,\ M = \text{const}), \qquad \{d_n\} \in l_2, \tag{3}$$

$$\{b_{i,n}\} \in l_2 \quad (i = 0, 1, \dots, p-1); \qquad 1 + C(t) \neq 0 \quad (|t| = 1).$$

We shall seek the solution of the system in l_2.

We write system (1) in the following form:

$$x_n + \sum_{k=-\infty}^{\infty} a_{n-k}x_k - d_n = \omega_n'\eta(n) \qquad (n = \dots, -1, 0, 1, \dots), \tag{4}$$

$$x_n + \sum_{k=-\infty}^{\infty} b_{n,k}x_k - d_n = 0 \qquad (n = 0, 1, \dots, p-1), \tag{5}$$

$$x_n + \sum_{k=-\infty}^{\infty} c_{n-k}x_k - d_n = \omega_n\eta(p-1-n) \qquad (n = \dots, -1, 0, 1, \dots), \tag{6}$$

---

* Laurent transformations are the relations

$$A(z) = \sum_{n=-\infty}^{\infty} a_n z^n \qquad (r < |z| < R)$$

and

$$a_n = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{A(z)}{z^{n+1}}\,dz \qquad (r < \rho < R,\ n = 0,\ \pm 1, \dots),$$

which connect the sequence $\{a_n\}$ and the function $A(z)$.

where $\eta(n) = 1$ for $n \geq 0$ and $\eta(n) = 0$ for $n < 0$.

Passing in relations (4) and (6) to images on the unit circle $\left(A(t) = \sum_{n=-\infty}^{\infty} a_{nt}^{n} \text{ for } |t| = 1\right)$, we arrive at the following problem:

Find the boundary values of the functions $\Omega_1^+(z)$ and $\Omega^-(z)$, analytic respectively inside and outside the unit circle, belonging on the contour to the class L_2 and satisfying the boundary condition

$$\Omega_1^+(t) = \frac{1 + A(t)}{1 + C(t)} \Omega^-(t) + \frac{1 + A(t)}{1 + C(t)} \left[D(t) + \sum_{n=0}^{p-1} \omega_{nt}^{n} \right] - D(t),$$

where

$$\Omega_1^+(t) = \sum_{n=0}^{\infty} \omega_n' t^n, \qquad \Omega^-(t) = \sum_{n=-\infty}^{-1} \omega_n t^n;$$

$\omega_0, \omega_1, \ldots, \omega_{p-1}$ must be such that equations (5) are satisfied for

$$x_k = \frac{1}{2\pi i} \int_{|t|=1} \frac{\Omega^-(t) + \sum_{n=0}^{p-1} \omega_{nt}^{n} + D(t)}{1 + C(t)} \frac{dt}{t^{k+1}} \qquad (k = 0, \pm 1, \ldots). \qquad (7)$$

In solving this problem, the solution of the Riemann problem is used (10). In view of the assumptions imposed on the coefficients of the infinite system, the function $\dfrac{1 + A(t)}{1 + C(t)}$ will satisfy the Hölder condition.* Formula (7) gives the required solution of system (1).

* The conditions imposed on the coefficients of the given system may be considerably weakened if one uses the results of I. B. Simonenko on the solution of the Riemann boundary-value problem with continuous coefficients (unpublished; report at the Fourth All-Union Conference on Function Theory).

Case II. Let $|a_n| < \dfrac{M}{|n|^{\lambda+1}\alpha^n}$, $|c_n| < \dfrac{M}{|n|^{\lambda+1}\beta^n}$ $(0 < \lambda \leq 1,\ \alpha \leq \beta,\ M = \text{const})$, and let the sequences $\{b_{i,n}\rho^n\}$ $(i = 0, 1, \ldots, p-1)$ and $\{d_n \rho^n\}$ belong to l_2 for every ρ in the interval $[\alpha, \beta]$. We shall seek the solution of the infinite system (1) in the class of sequences satisfying the condition $\{x_n \rho^n\} \in l_2$ for every ρ in the interval $[\alpha, \beta]$. This class is the widest possible among the admissible ones.

In this case, in terms of transforms, we obtain a boundary-value problem on a composite contour with the following conditions:

$$[1 + A(\zeta)] X(\zeta) - D(\zeta) = \Omega_1^+(\zeta), \qquad |\zeta| = \alpha,$$

$$[1 + C(\zeta)]X(\zeta) - D(\zeta) = \Omega^-(\zeta) + \sum_{n=0}^{p-1} \omega_n \zeta^n, \qquad |\zeta| = \beta,$$

where $X(\zeta)$, $\Omega_1^+(\zeta)$, and $\Omega^-(\zeta)$ must be boundary values of functions: $X(z)$, analytic in the annulus $\alpha < |z| < \beta$; $\Omega_1^+(z)$, analytic for $|z| < \alpha$; and $\Omega^-(z)$, analytic for $|z| > \beta$. As in the preceding case, $\omega_0, \omega_1, \dots, \omega_{p-1}$ may be used in satisfying (5).

The solution of the infinite system is obtained by passing to the original from the function $X(z)$.

§ 2. Consider the system (2), assuming that the conditions (3) are satisfied. We shall seek the solution $\{x_n\}$ in the class l_2.

Introducing, analogously to the preceding, ω_n and passing to transforms, we shall have

$$X(t) + A(t) \sum_{\nu=0}^{m-1} X[\omega_\nu(t)] - D(t) = \Omega^+(t),$$

$$X(t) + C(t) \sum_{\nu=0}^{m-1} X[\omega_\nu(t)] - D(t) = \Omega^-(t), \tag{8}$$

where $\omega_\nu(t)$, $\nu = 0, 1, \dots, m-1$, represent the rotation group ($\omega_\nu(t) = e^{2\pi i\nu/m} t$) and $|t| = 1$.

By eliminating $X[\omega_\nu(t)]$ ($\nu = 0, 1, \dots, m-1$) from relations (8), one can arrive at a Riemann boundary-value problem for automorphic functions with respect to

$$\sum_{\nu=0}^{m-1} \Omega^\pm[\omega_\nu(t)]$$

with the boundary condition

$$\sum_{\nu=0}^{m-1} \Omega^+[\omega_\nu(t)] = \frac{1 + \sum_{\nu=0}^{m-1} A[\omega_\nu(t)]}{1 + \sum_{\nu=0}^{m-1} C[\omega_\nu(t)]} \sum_{\nu=0}^{m-1} \Omega^-[\omega_\nu(t)] + \frac{\sum_{\nu=0}^{m-1} \{A[\omega_\nu(t)] - C[\omega_\nu(t)]\}}{1 + \sum_{\nu=0}^{m-1} C[\omega_\nu(t)]} \sum_{\nu=0}^{m-1} D[\omega_\nu(t)].$$

The quantities $\Omega^\pm(t)$ are found by solving the jump problem

$$\Omega^+(t) - \Omega^-(t) = [A(t) - C(t)] \frac{\sum_{\nu=0}^{m-1} \{\Omega^+[\omega_\nu(t)] + D[\omega_\nu(t)]\}}{1 + \sum_{\nu=0}^{m-1} A[\omega_\nu(t)]}.$$

The required function $X(t)$ is found from conditions (8). The solution of the given system is obtained by the formula

$$x = \frac{1}{2\pi i} \int_{|t|=1} \left(D(t) - \frac{[A(t)+C(t)] \sum_{\nu=0}^{n_1-1} \{\Omega^+[\omega_\nu(t)] + D[\omega_\nu(t)]\}}{2\left\{1 + \sum_{\nu=0}^{m-1} A[\omega_\nu(t)]\right\}} + \right.$$

$$\left. + \frac{1}{2\pi i} \int_{|\tau|=1} \frac{[A(\tau)-C(\tau)] \sum_{\nu=0}^{m-1} \{\Omega^+[\omega_\nu(\tau)] + D[\omega_\nu(\tau)]\}}{1 + \sum_{\nu=0}^{m-1} A[\omega_\nu(\tau)]} \frac{d\tau}{\tau - t} \right) \frac{dt}{t^{n+2}}$$

$$(n = 0,\ \pm 1, \ldots).$$

The method proposed above also makes it possible to study infinite systems of the following types:

$$x_n + \sum_{k=-\infty}^{\infty} \sum_{\nu=0}^{m-1} e^{2\nu k\pi i/m} a_{\nu,n-k} x_k = d_n \qquad (n < p_1),$$

$$x_n + \sum_{k=-\infty}^{\infty} b_{n,k} x_k = d_n \qquad (p_1 \le n \le p_2 - 1);$$

$$x_n + \sum_{k=-\infty}^{\infty} \sum_{\nu=0}^{m-1} e^{2\nu k\pi i/m} c_{\nu,n-k} x_k = d_n \qquad (p_2 \le n);$$

$$x_n + \sum_{k=-\infty}^{p_1-1} a_{n-k} x_k + \sum_{k=p_1}^{p_2-1} b_{n,k} x_k + \sum_{k=p_2}^{\infty} c_{n-k} x_k = d_n \qquad (n = \ldots, -1, 0, 1, \ldots).$$

In conclusion I express my deep gratitude to the supervisor of this work, Prof. F. D. Gakhov.

Rostov-on-Don
State University

Received
15 VII 1958

REFERENCES

1. N. Wiener, E. Hopf, Sitzungsber. Preuss. Akad. Wiss., Phys.-Math. Kl., 30–32, 696 (1931).
2. V. A. Fock, *Matem. sborn.*, 14(56), issue 1–2, 3 (1944).
3. I. M. Rapoport, *Sborn. tr. Inst. matem. AN USSR*, 12 (1949).
4. F. D. Gakhov, Yu. I. Cherskii, *Uch. zap. Kazansk. gos. univ.*, 114, 8, 21 (1954).
5. F. D. Gakhov, Yu. I. Cherskii, *Izv. AN SSSR, ser. matem.*, 20, No. 1, 33 (1956).
6. Yu. I. Cherskii, *Uch. zap. Kazansk. gos. univ.*, 113, 10, 43 (1953).
7. E. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, 1948.
8. F. D. Gakhov, L. I. Chibrikova, *Matem. sborn.*, 35, issue 3, 395 (1954).
9. V. S. Rogozhin, *DAN*, 114, No. 3, 488 (1957).
10. F. D. Gakhov, *Boundary-Value Problems*, 1958.
11. Ya. N. Feld, *DAN*, 102, No. 2, 257 (1955).