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Abstract

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ON LINEAR METHODS OF SUMMATION

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This note gives some results concerning the summation of sequences by Toeplitz matrices.

Definitions. A matrix A is **stronger** than B if every sequence summable by B to a finite limit is summable by A to the same limit. If this property also extends to sequences summable to infinity of one sign, we say that the matrix A is **completely stronger** than B. We shall denote the elements of the matrix A by a_{ij}, where the index i is the row number and j the column number. A matrix A is called **normal** if $a_{ij} = 0$ $(i < j)$ and $a_{ii} \neq 0$. A chain of matrices A_i will be called **embedded** if A_{i+1} is stronger than A_i. Analogously we define a **completely embedded** chain of matrices. We shall denote the norms of matrices by

$$|A| = \sup_i \sum_{j=1}^{\infty} |a_{ij}|; \qquad \|A\| = \lim_i \sum_{j=1}^{\infty} |a_{ij}|.$$

§ 1. **Theorem 1.** *Let an arbitrary countable set of Toeplitz matrices be given, and let there be given a functional series diverging on the set E, whose terms are bounded on this set. Then zeros can be inserted between the terms of this series in such a way that the resulting series is not summable at any point of the set E by any of the prescribed matrices.*

In the case when all the matrices coincide and the terms of the series are constants, we obtain results contained in ([1]).

Corollary 1. Consider a function $\Phi(x) \in L(0, 2\pi)$ with a Fourier series divergent everywhere (an example of A. N. Kolmogorov ([2])). It is known that a Fourier series is summable almost everywhere by the method $(C, 1)$ (the Fejér-Lebesgue theorem). At the same time, it follows from our theorem that the Fourier series of the function $\Phi(x)$ can be interspersed with zeros in such a way that it is nowhere summable by any of the methods (C, α).

Corollary 2. Consider the series with partial sums $1, 0, 1, 0, \ldots$. Applied to it, our theorem is a generalization of Steinhaus' s theorem ([3]) to the case of a countable set of Toeplitz matrices. Namely, we obtain the following assertion: *there exists a sequence consisting of ones and zeros which is not summable by* *any matrix from a previously prescribed countable set of Toeplitz matrices.* We note that this sequence can always be chosen so that it contains no triple of consecutive zeros or consecutive ones.

Let us point out that Theorem 1 can be used in proving certain theorems on unconditional summability.

§ 2. Corollary 2 shows that the sum of the fields of a countable set of Toeplitz matrices preserves essential features of the field of a single matrix. In this connection it is natural to pose the question of the coverability of a countable set of matrices, i.e., of the construction of a matrix stronger than a countable

set of data. It is known that in the most general form this question is answered in the negative (see, for example, ([4])). It is interesting, however, to find conditions under which the problem has a solution. We shall be interested, in particular, in the question of covering a chain of nested normal Toeplitz matrices. It is possible to prove the existence of a non-coverable chain of such matrices (in the space of bounded sequences this fact follows from the results of ([4])). Sufficient conditions for the coverability of such a chain will be obtained below as a consequence of more general theorems.

Theorem 2. *Let a countable set of arbitrary Toeplitz matrices C_i be given. Form new matrices $A_i = C_i \cdot C_{i-1} \cdots C_1 \cdot A_0$. Then, in order that there exist a Toeplitz matrix stronger than all A_i in the space of bounded sequences, it is sufficient that the inequality*

$$\sup_{i>j} \|C_i \cdots C_{j+2} \cdot C_{j+1}\| = H(j) < \infty. \tag{1}$$

be satisfied.

Theorem 2 contains the results of ([5]). The condition contained in ([5]) is the following:

$$\prod_{i=1}^{\infty} |C_i| < \infty. \tag{2}$$

Condition (1) is broader than (2), since

$$\|C_i \cdots C_{j+1}\| \le \prod_{k=j+1}^{i} \|C_k\| \le \prod_{k=j+1}^{i} |C_k| \le \prod_{k=1}^{\infty} |C_k|.$$

For arbitrary sequences an analogous theorem holds if the matrices C_i are finite-row. Namely, theorem 3 is valid.

Theorem 3. *Let a countable set of finite-row Toeplitz matrices C_i be given. Then, in order that there exist a matrix stronger than all $A_i = C_i \cdot C_{i-1} \cdots C_1 \cdot A_0$, it is sufficient that condition (1) be satisfied.*

We shall apply the last theorem to the question posed above concerning the covering of a nested chain of normal matrices. First we formulate a lemma.

Lemma. *Let A and B be two normal Toeplitz matrices, and let A be stronger than B. Then $A = CB$, where C is a Toeplitz matrix.*

It follows from the lemma that any chain of nested normal matrices can be obtained by the construction indicated in theorem 3. Proceeding from this, we obtain the following sufficient conditions.

Theorem 4. *Let A_i be a chain of nested normal Toeplitz matrices. If the condition $\sup_{i>j} \|C_{ij}\| = H(j) < \infty$ is satisfied, where $A_i = C_{ij}A_j$, then there exists a normal matrix stronger than all A_i.*

It follows, in particular, from theorem 4 that *there exists a matrix summing any prescribed countable set of sequences in advance.*

§ 3. Theorem 3 proves useful in considering questions analogous to those discussed in the preceding section, but concerning the complete coverability of a chain of completely nested matrices. It is interesting that this time the answer turns out to be essentially different than in the cases of bounded ([4]) and ordinary (theorem 4) coverability.

Theorem 5. *Let a countable set of completely regular finite-row matrices C_i be given. Form new matrices $A_i = C_1 \cdot C_{i-1} \cdots C_1 \cdot A_0$. Then there exists a matrix completely stronger than all A_i.*

From Theorem 5 it follows:

Theorem 6. *For any chain of fully nested normal Toeplitz matrices there exists a matrix fully stronger than all the matrices of the chain.*

We note the following theorem, close to this one in formulation but requiring a different proof.

Theorem 7. *For any chain of fully nested positive Toeplitz matrices (not necessarily finite-row) there exists a matrix which is boundedly stronger than all of them taken together.*

§ 4. In this paragraph we consider the question of summing a finite set of sequences to prescribed numbers. For bounded sequences it was studied most fully in the work of A. L. Brudno (⌢4). We consider this question for arbitrary sequences.

It is obvious that in the case of linear dependence of the sequences (sequences are called linearly dependent if there exists a nontrivial convergent linear combination of them) the generalized limits delivered by a Toeplitz matrix are subject to the same linear dependence. Suppose now that the given sequences are linearly independent. It turns out that in this case the problem always has a solution.

Theorem 8. *Let $m+n$ linearly independent sequences a_i^k $(1 \leqslant k \leqslant m+n;\ i = 1, 2, \ldots)$ and m numbers λ_k be given. Then there exists a normal matrix summing*

a_i^k $(1 \leqslant k \leqslant m)$, *respectively, to* λ_k, *and not summing* a_i^k $(m+1 \leqslant k \leqslant m+n)$.

In the case when all sequences are bounded, we obtain the corresponding results of A. L. Brudno (^4), and the proof can be carried out more simply.

We note that the results of this paragraph may be generalized in a certain way to the case of a countable set of sequences.

§ 5. **Definition.** The **kernel** of a bounded sequence is the minimal interval containing its closure.

In this paragraph we formulate one theorem on kernels (we note that it is related to the study of the question of complete coverability of two fully regular matrices).

We shall call two matrices **compatible at infinity** if there does not exist a sequence summed by one of them to $+\infty$, and by the other to $-\infty$.

Theorem 9. *Let* A *and* B *be two Toeplitz matrices compatible at infinity. Then, for any bounded sequence* t_i, *the kernels of the sequences* $A(t_i)$ *and* $B(t_i)$ *intersect.*

Corollary. Any two Toeplitz matrices compatible at infinity are boundedly compatible.

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References Cited

1. V. M. Darevskii, *Matem. sborn.*, **7** (49), 3, 549 (1940).

2. A. N. Kolmogorov, *C. R.*, **183**, 1327 (1926).

3. H. Steinhaus, *Prace mat.-fiz.*, **22**, 121 (1911).

4. A. L. Brudno, *Matem. sborn.*, **16** (58), 2, 205 (1945).

5. G. Peterson, *Math. Zs.*, **68**, 184 (1957).

Note: Figure translations are in progress. See original paper for figures.