RECOVERY OF TENSOR FORCES FROM SCATTERING DATA

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Abstract

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RECOVERY OF TENSOR FORCES FROM SCATTERING DATA

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1. In our note ([1]) an analysis and a method of solution were given for the inverse problem of scattering theory for a system of differential equations whose coefficients have a first absolute moment on the half-axis $0 < x < \infty$. Starting from these results, one can carry out an analogous investigation also in the case most interesting for physics, when the coefficients of the differential equations contain singularities of order x^{-2} (both at zero and at infinity).

We shall confine ourselves here to considering a system of two equations, equivalent to the matrix equation

$$Y'' - \left[V(x) + 6x^{-2}P\right]Y + \lambda^2 Y = 0 \qquad (0 < x < \infty), \tag{A}$$

where $V(x) = \|v_{jk}(x)\|_1^2$ is a Hermitian matrix,* satisfying, for some $\varepsilon > 0$, the condition

$$\int_0^\infty t^{1+\theta}|V(t)|\,dt < \infty \qquad (-\varepsilon < \theta < \varepsilon), \tag{1}$$

and

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

As is known ([2,3]), the Schrödinger equation for the deuteron reduces to such a system if the tensor forces of interaction are taken into account.

2. Let us consider the boundary-value problem generated by the system (A) and the boundary condition

$$Y(0) = 0. \tag{B}$$

Theorem 1. *The boundary-value problem (A)–(B) has a continuous spectrum on the entire positive half-axis ($\lambda^2 > 0$) and, possibly, a finite number of nonpositive eigenvalues $0 \geq \lambda_1^2 > \lambda_2^2 > \cdots > \lambda_p^2$. For λ^2 from the spectrum there exist solutions $U(x,\lambda)$ of equation (A), vanishing at zero for $x = 0$ and generating Parseval' s equality, equivalent to the following expansion of the δ-function:*

$$\delta(x-y)\cdot I = \sum_{k=1}^{p} U(x,\lambda_k)U^*(y,\lambda_k) + \frac{1}{2\pi}\int_0^\infty U(x,\lambda)U^*(y,\lambda)\,d\lambda, \qquad (2)$$

where I is the identity matrix, and U^* is the matrix Hermitian conjugate to U.

The matrices $U(x,\lambda)$ entering formula (2) can be normalized so that, as $x \to \infty$, the asymptotic equalities hold

$$U(x,\lambda) \sim e^{i\lambda x}\cdot I - e^{i\lambda x}\cdot S(-\lambda) \qquad (\lambda^2 > 0), \qquad (3')$$

$$U(x,\lambda_k) \sim e^{-|\lambda_k|x}\cdot M_k \qquad (\lambda_k^2 < 0), \qquad (3'')$$

* We note that all matrices occurring in the paper—$Y(x), U(x,\lambda), S(\lambda), M_k$, etc. —are square matrices of order two.

If $A = \|a_{jk}\|_2^1$, then, by definition,

$$|A| = \max\{|a_{11}| + |a_{12}|,\ |a_{21}| + |a_{22}|\}.$$

and, if $\lambda_1 = 0$, then

$$U(x,0) \sim x^{-2}\cdot M_1, \qquad M_1 = mP \quad (m > 0), \qquad (3''')$$

where $S(\lambda)$ is a unitary matrix, called the **scattering matrix**; M_k is a Hermitian matrix whose rank is equal to the multiplicity of the eigenvalue λ_k^2.

The collection of the scattering matrix $S(\lambda)$, the eigenvalues $\lambda_k^2 \leqslant 0$, and the corresponding matrices M_k will be called, for brevity, the **scattering data** of the boundary-value problem (A)–(B). Below a method is presented for reconstructing the potential $V(x)$ from the scattering data of problem (A)–(B), and the necessary and sufficient conditions that these data must satisfy are found.

3. The scattering matrix $S(\lambda)$ of problem (A)–(B) has the following properties:

1°. The matrix $I - S(\lambda)$ is the Fourier transform of a Hermitian matrix $F_1(t)$, so that

$$F_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[I - S(\lambda)]e^{i\lambda t}\,d\lambda. \tag{4}$$

The elements of the matrix $F_1(t)$, $-\infty < t < \infty$, can be represented as the sum of two functions, one of which is summable, while the other is square-summable and bounded; for all $t > 0$ there exists $F_1'(t)$ and

$$\int_0^{\infty} t^{1+\theta}\,|F_1'(t)|\,dt < \infty \quad (-\varepsilon < \theta < \varepsilon).$$

2°. $S(0)P = P$.

From property 1° of the matrix $S(\lambda)$, according to Theorem 4 of note ([1]), it follows that if we construct the matrix

$$F(t) = {\sum}' M_k^2 e^{-|\lambda_k| t} + F_1(t), \tag{5}$$

where the summation on the right-hand side is extended over all nonzero eigenvalues of problem (A)–(B), then the equation

$$F(x+y) + K(x,y) + \int_x^{\infty} K(x,t)F(t+y)\,dt = 0 \tag{6}$$

has, for all $x > 0$, a unique solution $K(x,y)$, with $K(x,x)$ differentiable for every $x > 0$, and the set of matrix solutions $Z(x,\lambda)$ of the equation

$$Z'' - W(x)Z + \lambda^2 Z = 0, \tag{7}$$

$$W(x) = -2\frac{d}{dx}K(x,x), \tag{8}$$

which are asymptotically equal as $x \to \infty$ to the right-hand sides of formulas ($3'$) and ($3'''$), gives an expansion of the delta function analogous to (2).

For the further characterization of the scattering data of problem (A)–(B), let us consider (as in ([1])) the equations

$$x(t) + \int_0^{\infty} x(\xi)F(t+\xi)\,d\xi = 0, \qquad 0 \leqslant t < \infty, \tag{I}$$

$$-y(t) + \int_{\infty}^{0} y(\xi)F_1(t+\xi)\,d\xi = 0, \qquad -\infty < t \leqslant 0, \tag{II}$$

$$z(t) + \int_0^\infty z(\xi) F_1(t+\xi)\, d\xi = 0, \qquad 0 \leqslant t < \infty. \tag{III}$$

and denote the number of their linearly independent vector-solutions, respectively, by n_1, n_2, and n_3, and the Fourier transform of any nonzero solution of equation (I) by $\tilde{x}(\lambda)$.

Let r denote the sum of the ranks of all matrices M_k (if $\lambda_1 = 0$, then the rank of the matrix $M_1 = mP$ is included in r).

Then the following four cases are possible:

a) all $\lambda_k \neq 0$, $n_1 = 0$, $n_2 = 0$, $n_3 = r$, the matrix $K(x, y)$ (see (6)) satisfies the equality

$$P \left\{ I + \int_0^\infty K(0,t)\, dt \right\} = \gamma P,$$

where γ is a nonzero number;

b) $\lambda_1 = 0$, $M_1 = mP$, $n_1 = 1$, $n_2 = 0$, $n_3 = r$, $\lambda^2 \tilde{x}(\lambda)(I-P) \to 0$ as $\lambda \to \infty$;

c) all $\lambda_k \neq 0$, $n_1 = 1$, $n_2 = 0$, $n_3 = r$, $\lambda^2 \tilde{x}(\lambda)(I-P) \to 0$ as $\lambda \to \infty$, and, if $S(0) = I$, then $\tilde{x}(0)(I-P) \neq 0$;

d) all $\lambda_k \neq 0$, $n_1 = 1$, $n_2 = 1$, $n_3 = r+1$, $\lambda^2 \tilde{x}(\lambda)(I-P) \to 0$ as $\lambda \to \infty$, and, if $S(0) = I$, then $\tilde{x}(0)(I-P) \neq 0$.

The listed properties are characteristic for the scattering data of problem (A)–(B), i.e., the following theorem is valid.

Theorem 2. In order that the given unitary matrix $S(\lambda)$, the numbers $\lambda_k^2 \leq 0$, and the Hermitian matrices M_k $(k = 1, 2, \dots, p)$ be the scattering data of problem (A)–(B) with a Hermitian potential $V(x)$ satisfying condition (1), it is necessary and sufficient that conditions 1°, 2°, and one of the conditions a), b), c), or d) be fulfilled.

Moreover, if $S(0) = I$, then the potential $V(x)$ is determined uniquely by the scattering data. If, however, $S(0) \neq I$ (i.e., $\lambda = 0$ is a virtual level), then the potential $V(x)$ is determined uniquely by the scattering data only in case a), while in cases b), c), and d) there exists a one-parameter family of potentials $V(x)$ satisfying condition (1) and such that the corresponding boundary-value problems (A)–(B) have identical scattering data.

4. To reconstruct the potential $V(x)$ of problem (A)–(B) (in the case of non-uniqueness–the whole family of potentials) from the scattering data, we use the relation between $V(x)$ and the matrix $W(x)$ constructed by formula (8) from the solution of equation (6).

The characteristic properties of the scattering data (see Theorem 2) ensure the existence, for $\lambda = 0$, of a solution $Z(x) = \|z_{jh}(x)\|_1^2$ of equation (7) such that* $Z(x)P = Z(x)$ and

1) $Z(x) = -3xP + O(1)$ as $x \to \infty$;
2) $Z(0) = 0$ and $(I - P)Z'(0) = 0$ in case a), $Z(0) = 0$ in case b), $PZ(0) = 0$, and $(I - P)Z(0) \neq 0$ in cases c) and d). Moreover, in cases c) and d)

$$\lim_{x \to 0} [Z^*(x)Z(x, \lambda) - Z^*(x)Z'(x, \lambda)] = \lambda^2 h^2 Z^*(0)Z(0, \lambda), \tag{9}$$

where $Z(x, \lambda)$ is the solution of equation (7) with asymptotic (3′) as $x \to \infty$, and $h^2 > 0$ and does not depend on λ. Note that in the cases of non-uniqueness the matrix $Z(x)$ contains one arbitrary parameter.

In all cases the desired potential $V(x)$ is found by the formula

$$V(x) = W(x) + 2\frac{d}{dx}[Z(x)Z_1^*(x)] - 6x^{-2}P, \tag{10}$$

* If

$$\int_1^\infty t^2 |F_1'(t)|\, dt < \infty,$$

then

$$Z(x) = \begin{pmatrix} 0 & \alpha \\ 0 & -3x + \beta \end{pmatrix} \int_x^\infty K(x, t) \begin{pmatrix} 0 & \alpha \\ 0 & -3t + \beta \end{pmatrix} dt,$$

where α and β are determined from the conditions at the point $x = 0$.

where

$$Z_1(x) = Z(x) \left[I - (1 + c^2)P - \int_0^x Z^*(t)Z(t)\, dt \right]^{-1} \tag{11}$$

and $c^2 = 0$ in case a), $c^2 = m^2$ in case b), $c^2 = h^2|z_{12}(0)|^2$ in cases c) and d) (see formula (9)).

5. To illustrate the results obtained, we give several examples.

I (case a)). $S(\lambda) = (\lambda + i)(\lambda - i)^{-1}(I - P) + P$, the point spectrum is absent. Then $V(x) = W(x) = -2(\operatorname{ch} x)^{-2}(I - P)$.

II (case b)). $S(\lambda) = (\lambda + i)(\lambda - i)^{-1}(I - P) + (\lambda + i)(\lambda + 2i)[(\lambda - i) \times (\lambda - 2i)]^{-1}P$, $\lambda_1^2 = 0$, $M_1 = mP$ $(m > 0)$. Then $W(x) = -2(\operatorname{ch} x)^{-2} \times (I - P) + 6(\operatorname{sh} x)^{-2}P$,

$$Z(x) = \begin{pmatrix} 0 & \alpha \operatorname{th} x \\ 0 & \varphi(x) \end{pmatrix},$$

where α is an arbitrary number and

$$\varphi(x) = -3x + 4.5 - 2.25\,(e^{-2x} - 1 + 2x)(\operatorname{sh} x)^{-2};$$

the potential $V(x)$ is expressed in terms of $W(x)$ and $Z(x)$ by formulas (10) and (11).

III (case c)). $S(\lambda)$ is the same as in example II, $\lambda_1^2 = -1$, $M_1 = I - P$. Then $W(x) = -12e^{-2x}(1 + 1.5e^{-2x})^{-2}(I - P) + 6(\operatorname{sh} x)^{-2}P$,

$$Z(x) = \begin{pmatrix} 0 & \alpha(1 - 1.5e^{-2x})(1 + 1.5e^{-2x})^{-1} \\ 0 & \varphi(x) \end{pmatrix}$$

and $c^2 = 0.2\alpha^2$.

IV (case d)). $S(\lambda) = (\lambda - i)(\lambda + i)^{-1}(I - P) + (\lambda + i)(\lambda + 2i)[(\lambda - i)(\lambda - 2i)]^{-1}P$, the point spectrum is absent. Then $W(x) = 6(\operatorname{sh} x)^{-2}P$,

$$Z(x) = \begin{pmatrix} 0 & \alpha \\ 0 & \varphi(x) \end{pmatrix}$$

and $c^2 = \alpha^2$.

In examples III and IV, α is arbitrary, and the function $\varphi(x)$ is the same as in example II. We note that in all the examples given, $S(0) \neq I$; in the first example $V(x)$ is determined uniquely, while in the last three it is not unique, since the arbitrary parameter α contained in $Z(x)$ obviously also enters into expression (10) for $V(x)$.

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Note: Figure translations are in progress. See original paper for figures.