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Abstract

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ON A PROBLEM OF UNIFORM DISTRIBUTION OF A SYSTEM OF FUNCTIONS

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In the present article, by the method of N. M. Korobov, we generalize multidimensional problems on the uniform distribution of systems of functions that are products of an exponential function in many variables, which were considered by him in the paper ([1]). The proof of the theorem formulated below is based on a lemma that is an analogue of the main lemma of paper ([1]).

Let s finite-difference equations be given:

$$\psi_\nu(x) = a_1^{(\nu)}\psi_\nu(x-1) + \dots + a_{n_\nu}^{(\nu)}\psi_\nu(x-n_\nu), \quad \nu = 1, 2, \dots, s, \tag{1}$$

with integer coefficients; $a_{n_\nu}^{(\nu)} \neq 0$, $n_\nu \geq 1$ for each ν. Choose s prime numbers $p_1 < p_2 < \dots < p_s$ such that

$$p_1 > \max_{1\leq\nu\leq s} \left|a_{n_\nu}^{(\nu)}\right|.$$

We assign initial values of the function $\psi_\nu(x)$ by putting $\psi_\nu(j) = \delta_j^{(\nu)}$, where $\delta_j^{(\nu)}$ is an integer and $0 \leq \delta_j^{(\nu)} \leq p_\nu - 1$, $j = 1, 2, \dots, n_\nu$, $\nu = 1, 2, \dots, s$, and at least one of the numbers $\delta_1^{(\nu)}, \dots, \delta_{n_\nu}^{(\nu)}$ is different from zero for each ν. Denote by $\lambda_{\nu 1}, \dots, \lambda_{\nu n_\nu}$ the roots of the characteristic equation

$$\lambda^{n_\nu} = a_1^{(\nu)}\lambda^{n_\nu - 1} + \dots + a_{n_\nu}^{(\nu)}$$

of the finite-difference equation (1).

We shall assume that for each ν the roots $\lambda_{\nu 1}, \lambda_{\nu 2}, \dots, \lambda_{\nu n_\nu}$ are distinct and that, for the quantities $\lambda_\nu = \lambda_{\nu 1}$ and $\theta_\nu = \max_{2\leq j\leq n_\nu} |\lambda_{\nu j}|$, the inequalities

$$\lambda_\nu > 1, \quad \theta_\nu < 1 \quad \text{for } \nu = 1, 2, \dots, s.$$

hold. Thus, for $n_\nu = 1$, λ_ν is an integer; for $n_\nu \geq 2$, λ_ν is a Pisot number.

Introduce the following notation:

1) τ_ν is any integer satisfying the conditions

$$\psi_\nu(x+\tau_\nu) \equiv \psi_\nu(x) \pmod{p_\nu}, \quad \tau_\nu \equiv 0 \pmod{p_\nu}, \quad \nu = 1, 2, \dots, s.$$

(Such a τ_ν can always be found according to Lemma 1 of paper ([1]).)

2) $f_\nu(x) = b_0^{(\nu)} + b_1^{(\nu)} x + \dots + b_{k_\nu}^{(\nu)} x^{k_\nu}$ is an integer polynomial of degree k_ν, not identically equal to zero modulo mod p_ν, $\nu = 1, 2, \dots, s$.

According to Lemma 2 of [1], choose the initial values of the functions $\psi_\nu(x)$, $\nu = 1, 2, \dots, s$, so that the congruence

$$\psi_\nu(z) \equiv 0 \pmod{p_\nu}, \qquad z = 1, 2, \dots, \tau_\nu,$$

has no more than τ_ν / p_ν solutions.

Under these conventions and notations the following lemma is valid.

Lemma. *For any integers $a \geq 0$, $r \geq 1$, the estimate holds*

$$\sum_{x=1}^{rp_1 \dots p_s \tau_1 \dots \tau_s} \exp 2\pi i \sum_{\nu=1}^{s} \frac{\psi_\nu(x) f_\nu(a+x)}{p_\nu(\lambda_\nu^{\tau_1 \dots \tau_s} - 1)} =$$

$$= O(rp_1 \dots p_{s-1} \tau_1 \dots \tau_s + rp_1 \dots p_s \ln(a + r\tau_1 \dots \tau_s)),$$

where the constant occurring in the symbol O depends on the quantities

$$a_1^{(1)}, \dots, a_{n_s}^{(s)}, \quad b_0^{(1)}, \dots, b_{k_s}^{(s)}, \quad n_1, \dots, n_s, \quad s.$$

Let now an infinite sequence of prime numbers be given,

$$p_1 < p_2 < p_3 < \dots,$$

whose growth is restricted by the requirement

$$p_{\nu+1} = O(p_\nu).$$

Denote by $\tau_{j1} < \tau_{j2} < \dots$ positive integers satisfying the conditions:

$$\psi_{j\nu}(x + \tau_{j\nu}) \equiv \psi_{j\nu}(x) \pmod{p_\nu}, \qquad \tau_{j\nu} \equiv 0 \pmod{p_\nu},$$

$$j = 1, 2, \dots, s, \qquad \nu = 1, 2, \dots,$$

where $\psi_{j\nu}(x)$ is a solution of the finite-difference equation

$$\psi(x) = a_1^{(j)}\psi(x-1) + \dots + a_{n_j}^{(j)}\psi(x-n_j), \qquad n_j \geq 1, \quad p_1 > \max_{1\leq j\leq s}|a_{n_j}^{(j)}|,$$

having the property that the number of solutions of the congruence

$$\psi_{j\nu}(z) \equiv 0 \pmod{p_\nu}, \qquad z = 1, 2, \dots, \tau_{j\nu},$$

does not exceed $\tau_{j\nu}/p_\nu$.

We require, in addition, that the estimate

$$\ln(\tau_{1(\nu+1)}, \dots, \tau_{s(\nu+s)}) = o(\tau_{1\nu} \dots \tau_{s(\nu+s-1)})$$

hold.

Let, further, $t_1 < t_2 < \dots$ be arbitrary integers such that

$$t_\nu \geq \tau_{1(\nu+1)} \dots \tau_{s(\nu+s)}, \qquad \ln t_\nu = O(\ln(\tau_{1(\nu+1)} \dots \tau_{s(\nu+s)}));$$

the integers $n_1, n_2, \dots$ are defined by the relation

$$n_{\nu+1} = n_\nu + t_\nu p_\nu \dots p_{\nu+s-1}\tau_{1\nu} \dots \tau_{s(\nu+s-1)}, \qquad n_1 = 0;$$

$\varphi(\nu) = o(p_\nu)$ is any integer-valued function, different from zero for sufficiently large values of the argument.

We assume that $\lambda_{j1}, \dots, \lambda_{jn_j}$ are the roots of the characteristic equation

$$\lambda^{n_j} = a_1^{(j)}\lambda^{n_j-1} + \dots + a_{n_j}^{(j)}, \qquad j = 1, 2, \dots, s,$$

that they are all distinct and

$$\lambda_j = \lambda_{j1} > 1, \qquad \theta_j = \max_{1\leq k\leq n_j} |\lambda_{jk}| < 1.$$

As in paper (1), it is easy to see that

$$\psi_{j(\nu+j-1)}(x) = \gamma_{j(\nu+j-1)}\lambda_j^x + O\left(p_{\nu+j-1}\theta_j^x\right), \qquad \gamma_{j(\nu+j-1)} = O\left(p_{\nu+j-1}\right),$$

$$j = 1, 2, \dots, s, \quad \nu = 1, 2, \dots$$

Define the numbers α_j by the series

$$\alpha_j = \sum_{i=1}^{\infty} \frac{\varphi(i)\gamma_{j(\nu+j-1)}}{p_{i+j-1}\left(\lambda_j^{\tau_1 i \cdots \tau_s (i+s-1)} - 1\right)} \left(\frac{1}{\lambda_j^{n_i}} - \frac{1}{\lambda_j^{n_i+1}}\right), \qquad j = 1, 2, \dots, s.$$

Theorem. Let $f_1(x), \dots, f_s(x)$ be integer polynomials, not identically equal to zero. Then the system of functions

$$\alpha_1 \lambda_1^x f_1(x), \dots, \alpha_s \lambda_s^x f_s(x)$$

is uniformly distributed in s-dimensional space.

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REFERENCES

1. N. M. Korobov, *Izv. AN SSSR, Ser. Mat.*, **17**, No. 5 (1953).

Note: Figure translations are in progress. See original paper for figures.