On the Function $\zeta(s)$

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Abstract

Full Text

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I have found a new estimate for $\zeta(1+it)$:

$$\zeta(1+it) = O\left((\ln t \, \ln\ln t)^{2/3}\right) \tag{1}$$

($t \geqslant t_0$, where t_0 is a sufficiently large constant > 1).
This estimate is a consequence of a new estimate for the sum

$$S = \sum_{a<x\leq b} e^{2\pi i f(x)}; \qquad f(x) = -\frac{t \ln x}{2\pi},$$

where, for an integer $n \geqslant 7$, the conditions

$$a < b \leqslant 2a, \qquad t = a^{n-\theta}, \qquad 0 < \theta \leqslant 1$$

are satisfied.

Namely, for $a > (4n)^{16n^2}$, for the sum S I obtained an estimate which can be put in the form (the letters $c, c_1, c_2, \ldots$ denote absolute positive constants)

$$|S| \leqslant e^{c \ln^2 n} a^{1-\frac{c_1}{n^2 \ln n}}.$$

Estimate (1), and other estimates analogous to it obtained by the same method, entail an improvement of a number of generally known results in the theory of the distribution of prime numbers. For example: $\zeta(s) = \zeta(\sigma + it)$ has no zeros in the region

$$\sigma \geqslant 1 - \frac{c_2}{(\ln t \, \ln\ln t)^{2/3}}.$$

Correspondingly, the remainder term in the asymptotic formula for the number $\pi(N)$ of primes not exceeding N is also improved, and so on. The method I have applied to derive the estimate for the sum S is a certain development of my earlier method. It can also be successfully applied to the derivation of new estimates for trigonometric sums belonging to broader classes. A detailed exposition of my new results will be given elsewhere.

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Note: Figure translations are in progress. See original paper for figures.