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Abstract

Full Text

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ON THE DEPENDENCE BETWEEN THE NUMBER OF CHANGES OF SIGN OF A SOLUTION OF A PARABOLIC EQUATION AND THE GROWTH OF THE SOLUTION

(Presented by Academician I. G. Petrovsky, 8 VII 1958)

Let there be given the equation

$$\frac{\partial u}{\partial t} = \sum_{i,k=1}^{n} a_{i,k}(t, x_1, \dots, x_n)\frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{n} b_i(t, x_1, \dots, x_n)\frac{\partial u}{\partial x_i} + c(t, x_1, \dots, x_n)u \equiv Lu, \tag{1}$$

defined in a domain D_T lying in the cylinder $0 < t < T$, $|x| < 1$, where $x = (x_1, \dots, x_n)$, and having limit points on each of the hyperplanes $t = 0$ and $t = T$. Suppose that the coefficients of equation (1) satisfy the following conditions: the coefficients $a_{i,k}$ are twice continuously differentiable, the coefficients b_i are continuously differentiable, and the inequalities

$$|a_{i,k}| < M, \qquad \left|\frac{\partial a_{ik}}{\partial x_j}\right| < M, \qquad \left|\frac{\partial^2 a_{ik}}{\partial x_j \partial x_l}\right| < M, \qquad |b_i| < M, \qquad \left|\frac{\partial b_i}{\partial x_i}\right| < M,$$

$$-M < C < 0, \qquad i, k = 1, \dots, n, \quad j, l = 1, \dots, n; \tag{2}$$

are satisfied, and

$$\sum_{i,k=1}^{n} a_{i,k}\xi_i\xi_k \Big/ \sum_{i=1}^{n} \xi_i^2 > a > 0. \tag{3}$$

Denote by S_T that part of the boundary of the domain D_T which is situated outside the hyperplanes $t = 0$ and $t = T$. Let $u(t, x)$ be a solution of equation (1), defined in D_T, continuous in $\overline{D_T}$, and vanishing on S_T. Denote by D_T^+ and D_T^- the sets of points $(t, x) \in D_T$ at which, respectively, $u(t, x) > 0$ and $u(t, x) < 0$. We shall call a component of the set D_T^+ or D_T^- *essential* if it has limit points on each of the hyperplanes $t = 0$ and $t = T$. The total number of all essential components of the sets D_T^+ and D_T^- will be called the *number of changes of sign* of the function $u(t, x)$ in D_T.

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Theorem. There exists a constant C, depending on the constant M in inequalities (2) and on the constant a in inequality (3), such that for every solution $u(t,x)$ of equation (1), defined in the domain D_T, $T \geq 1$, continuous in $\overline{D_T}$, and having in D_T N changes of sign, the inequality

$$\max_x |u(T,x)| \big/ \max_x |u(0,x)| < e^{-TN^{2/n}/C}. \tag{4}$$

holds.

This theorem is adjacent to the theorems of note ([1]), in which a similar result was formulated for solutions of elliptic equations. The proof of the theorem is based on lemmas likewise similar to the lemmas of note ([1]).

Lemma 1. There exists a constant C_1, depending on the constant M in inequality (2) and on the constant a in inequality (3), such that, under the condition

$$\mu_{n+1} D_T \leq T^{n/2+1}/C_1$$

(where by $\mu_k E$ we shall denote the k-dimensional measure of the set E), every solution $u(t,x)$ of equation (1), positive in D_T, continuous in $\overline{D}_T$, and vanishing on $\dot{S}_T$, satisfies the inequality

$$\max_x u(T,x) / \max_x u(0,x) < {}^1\!/_2.$$

The idea of the proof of this lemma is as follows. One can find a subdomain g of the domain D_T having the following properties:

1) g is situated between the hyperplanes $t = t_0$ and $t = t_0 + h^2$, $0 < h < 1$, and has limit points on each of them. Denote by σ that part of the boundary of the domain g which is situated strictly between the hyperplanes $t = t_0$ and $t = t_0 + h^2$, and by γ the remaining part of the boundary of g. For each t_1, $t_0 < t_1 < t_0 + h^2$, denote by σ_{t_1} the intersection of σ with the hyperplane $t = t_1$.

2) $\mu_{n+1} g < L_1 h^{n+1}$ and $\mu_n \gamma < L_2 h^n$, where L_1 and L_2 are absolute constants.

3) $u|_\sigma = \text{const} = u_0$ and $\partial u(t_1,x)/\partial n_{t_1}|_\sigma > 0$ for almost all $t_1 \in [t_0, t_0 + h^2]$, where $\partial u(t_1,x)/\partial n_{t_1}$ is the derivative in the direction of the inner normal to σ_{t_1} in the hyperplane $t = t_1$.

4)

$$\int_\sigma \frac{\partial u(t,x)}{\partial n_t}\, d\sigma > C_1 h^n \max_x u(T,x)/L_3,$$

where L_3 is an absolute constant.

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Applying to the left-hand side of the equality

$$\int_g \left(\frac{\partial u}{\partial t} - Lu \right) dt\, dx_1 \dots dx_n = 0$$

Green' s formula, we obtain

$$\int_{\gamma+\sigma} u\, dx_1 \dots dx_n - \int_\sigma \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_j}\, dt\, dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_n +$$

$$+ \int_\sigma \left(\sum_{i,j} \frac{\partial a_{ij}}{\partial x_i} - \sum_j b_j \right) u\, dt\, dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_n +$$

$$+ \int_g \left(- \sum_{i,j} \frac{\partial^2 a_{ij}}{\partial x_i \partial x_j} + \sum_i \frac{\partial b_i}{\partial x_i} - C \right) u\, dt\, dx_1 \dots dx_n = 0.$$

Applying inequalities (2), (3) and taking into account properties 1)–4) of the domain g, we find

$$aC_1 h^n \max_x u(T,x)/L_3 < (L_1 + (2n^2 + 2n + 1)L_2 Mh) h^n \max_x u(0,x).$$

It follows that, for sufficiently large C_1, the inequality

$$\max_x u(T,x) / \max_x u(0,x) < {}^1/_2$$

holds.

Lemma 2. Let P be the cylinder

$$t^2 + \sum_{i=2}^n x_i^2 < h^2, \qquad 0 < h < 1.$$

Let Γ_1 and Γ_2 be n-dimensional manifolds lying inside the cylinder P, with boundary on the boundary of P, each of which separates in the cylinder P the points with sufficiently large positive coordinates x_1 from the points

with sufficiently large, in absolute value, negative coordinates x_1. Suppose that for the points $(t,x) \in \Gamma_1 + \Gamma_2$ the inequality $|x| < 1$ is satisfied. Let G be the part of P lying between Γ_1 and Γ_2. Suppose that in G equation (1) is defined, satisfying in G the conditions (2) and (3). Let $u(t,x)$ be a solution of this equation, defined in G and continuously differentiable in $\bar{G}$. Suppose that $u(t,x)$ satisfies the following conditions: 1) $u|_{\Gamma_1} = u_0 > 0$, 2) $u|_{\Gamma_2} = 0$, 3)

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$\partial u/\partial n|_{\Gamma_2} \leq 0$, where $\partial/\partial n$ denotes differentiation along the inner normal, 4) $u(t,x) \leq -u_0$ for $x \in G$. Denote by G_1 the set of points $(t,x) \in G$ for which $u_0/2 < u(t,x) < u_0$. Then

$$\mu_{n+1} G_1 > h^{n+1}/C_2,$$

where C_2 is a constant depending on the constant M of inequalities (2) and on the constant a of inequality (3).

The idea of the proof of this lemma has common features with the idea of the proof of Lemma 1 of the present note and Lemma 2 of note (1). It is as follows. Put $h_1^{n+1}/\mu_{n+1} G_1 = C_2^*$. Let E_τ be the level set τ in G. There exists such a u^*, $u_0/2 < u^* < u_0$, that: a) the level set E_{u^*} contains no points (t,x) where $\operatorname{grad} u(t,x) = 0$, and, consequently, consists of smooth n-dimensional manifolds; b) the inequality

$$\int_{E_{u^*}} \left[\sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2 \right]^{1/2} d\sigma > \frac{C_2^* u_0 h^{n-1}}{M_1}, \tag{5}$$

holds, where M_1 is an absolute constant. Denote by G_{u^*} the set of points $(t,x) \in G$ for which $u(t,x) < u^*$. Applying Green' s formula to the left-hand side of the equality

$$\int_{G_{u^*}} \left(\frac{\partial u}{\partial t} - Lu \right) dt\, dx_1 \dots dx_n = 0$$

we obtain

$$\int_{\bar{\sigma}_{u^*} - G_{u^*}} u\, dx_1 \dots dx_n - \int_{E_{u^*} + \Gamma_2} \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_j}\, dt\, dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_n +$$

$$+ \int_{E_{u^*} + \Gamma_2} \left(\sum_{i,j} \frac{\partial a_{ij}}{\partial x_i} - \sum_j b_j \right) u\, dt\, dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_n +$$

$$+ \int_{G_{u^*}} \left(-\sum_{i,j} \frac{\partial^2 a_{ij}}{\partial x_i \partial x_j} + \sum_j \frac{\partial b_i}{\partial x_j} - c \right) u\, dt\, dx_1 \dots dx_n = 0.$$

Hence, with the aid of inequalities (5), (2), (3) and conditions 3) and 4), we obtain

$$a C_2^* u_0 h^n / M_1 < M \cdot M_2 u_0 h^h,$$

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where M_2 is an absolute constant. The last inequality gives us an estimate of C_2^* in terms of M and a.

With the aid of Lemmas 1 and 2, Lemma 3 is proved, analogous to Lemma 3 of note (1).

Lemma 3. Suppose that in the domain D_1 a solution $u(t,x)$ of equation (1) is defined, continuous on $\bar{D}_1$ and vanishing on the lateral surface S_1 of the domain D_1. Suppose that $u(t,x)$ has N changes of sign in D_1. Let $g_1, \dots, g_N$ be the essential components of the sets D_1^+ and D_1^-. Put

$$m' = \max_x |u(0,x)|, \qquad m'' = \max_x |u(1,x)|, \qquad m_i'' = \max_{(T,x)\in \bar{g}_i} |u(T,x)|.$$

There exists a constant C_3, depending on the constant M in inequality (2) and on the constant a in inequality (3), such that from the fact that

$$m''/m' > 2^{-N^2/n/C_3}, \tag{6}$$

it follows that

$$\min_i m_i'' > 2^{-N^2/n/[4(C_1\omega_n)^{2/n}]}, \tag{7}$$

where C_1 is the constant of Lemma 1 and ω_n is the volume of the unit n-dimensional ball.

From Lemmas 1 and 3 it follows:

Lemma 4. In the notation of the preceding lemma, the following assertion holds: there exists a constant C_4, depending on the constant M in inequality (2) and on the constant a in inequality (3), such that

$$m''/m' < 2^{-N^2/n/C_4}.$$

Proof. Suppose that inequality (6) is satisfied and, consequently, by Lemma 3, inequality (7) is satisfied. We have $\sum_{i=1}^N \mu_{n+1} g_i \leqslant \omega_n$, where ω_n is the volume of the unit n-dimensional ball. Therefore there exists an i_0 such that

$$\mu_{n+1} g_{i_0} \leqslant \omega_n/N. \tag{8}$$

Set $N_1 = [(N/2C_1\omega_n)^{2/n}]$. Suppose that $N_1 > 3$ (in the case $N_1 \leqslant 3$, the assertion of the lemma is easily obtained by constructing a barrier). Put $t_k = k/N_1$. Let

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$$m_{i_0,k} = \max_{(t_k,x)\in g_{i_0}} |u(t_k,x)|.$$

Let these maxima be attained at points P_k, $k = 0, 1, \dots, N_1$. Denote by H_k, $k = 1, \dots, N_1$, the layer $t_{k-1} < t < t_k$, and by $g_{i_0,k}$ the component of the intersection of g_{i_0} with this layer whose closure contains the point P_k. By the maximum principle, $g_{i_0,k}$ has boundary points on both boundaries of the layer H_k. Consider those k, $k = 1, 2, \dots, N_1$, for which

$$\mu_n g_{i_0,k} \leqslant 1/C_1 N_1^{n/2+1}. \tag{9}$$

The number of such k does not exceed $N_1/2$, since otherwise we would have

$$\mu_{n+1} g_{i_0} > \frac{N_1}{2}\,\frac{1}{C_1 N_1^{n/2+1}} = \frac{1}{2C_1 N_1^{n/2}} > \frac{\omega_n}{N},$$

which contradicts (8). Consequently, there are at least

$$N_1/2 - 1 > N^{2/n}/2(2C_1\omega_n)^{2/n}$$

distinct k for which the inequality opposite to (9) holds, i.e., such that Lemma 1 is applicable to $g_{i_0,k}$. From this lemma and the maximum principle we obtain that for such k the inequality $m_{i_0,k} < m_{i_0,k-1}/2$ holds. Hence

$$m'' > 2^{N^{2/n}/[4(2C_1\omega_n)^{2/n}]} \min_i m''_i \geqslant 2^{N^{2/n}/[4(2C_1\omega_n)^{2/n}]} m_{i_0} >$$

$$> 2^{N^{2/n}/[4(2C_1\omega_n)^{2/n}]} 2^{-N^{2/n}/[2(2C_1\omega_n)^{2/n}]} = 2^{-N^2/n/[4(2C_1\omega_n)^{2/n}]}.$$

It remains to put $C_4 = \max(C_3,\ 4(2C_1\omega_n)^{2/n})$.

The theorem formulated above is a simple consequence of Lemma 4.

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References

1. E. M. Landis, *DAN*, **123**, No. 4 (1958).

Note: Figure translations are in progress. See original paper for figures.

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